The geometry of the zeros of a polynomial in a complex variable. By Morris Marden. (Mathematical Surveys, no. 3.) New York, American Mathematical Society, 1949. 10+183 pp. $5.00.

Since the time of Gauss there has been continued interest in those problems that center around the location of the zeros of a polynomial. Short expositions of the theory have been given, and a lengthier one by Dieudonné, but now, with the publication of this work, the third in the series of Mathematical Surveys of the American Mathematical Society, a comprehensive treatise on the subject has been made available. The lengthy, 20-page bibliography attests to the aim of the author to cover the material thoroughly; and considerable work must have been involved to integrate this heterogeneous material into such unity as is possible. Even so, the book could not have been kept to 161 pages had the author not adopted the plan of putting considerable theoretical material into the problems that appear at the end of most sections. Where necessary, hints are given with the problems, a wise gesture as readers of the book will see.

In a well-written preface there is a brief account of the origin of the subject, and a statement of the two central problems that the book is to treat, together with a declaration of the principal working tools. The zeros of a polynomial $f(z)$ are functions of the coefficients. Thus one problem is to specify regions, determined by these coefficients, in which the zeros lie. Again, with $f(z)$ one may associate a second polynomial (frequently this is the derivative $f'(z)$), and thus arises the problem of relating the location of the zeros of the associate to the location of those of $f(z)$. There are also variants of these problems, of which something will be said as the individual chapters are discussed. As for the methods and tools that are used, they are classical. Thus, among other theorems, those of Cauchy, Rouché, and Hurwitz on zeros of analytic functions are of considerable use; and many theorems involve, either in proof or in statement, geometric and algebraic concepts of an elementary nature.

Chapter I is introductory. The above-mentioned classical theorems are stated and proofs given. Then various interpretations, from physics, geometry, and function-theory, are given for the zeros of the rational function

$$F(z) = \sum_{j=1}^{p} \frac{m_j}{z - z_j}.$$

One from physics goes back to Gauss: The zeros of $F(z)$ are the equilibrium points in a force field due to p masses $m_1, \cdots, m_p$ at the points $z_1, \cdots, z_p$ with the law of inverse distance. When the m_j are positive integers, $F(z)=0$ gives those zeros of $f'(z)$ that are not zeros of $f(z)$ itself, where $f(z)=\prod_{j=1}^{p}(z-z_j)^{m_j}$.

This leads at once to the Lucas theorem of Chapter II: Any convex polygon (or any circle) that contains the zeros of a polynomial $f(z)$ also contains the zeros of $f'(z)$. In this theorem the coefficients of $f(z)$ are free to lie anywhere in the complex plane; consequently one may expect to narrow the location of the roots of $f'(z)$ if one specializes the coefficients. Thus, if $f(z)$ is a *real* polynomial, the non-real zeros of $f'(z)$ are contained in the Jensen circles (theorem of Jensen). As the Lucas and Jensen theorems concern the zeros of the function $F(z)$ above, it is natural to consider a generalization to the zeros of the function $G(z)=\sum_{j=1}^{n} m_j f_j(z)$, where each $f_j(z)$ is a rational function with p finite zeros and q finite poles, and the m_j are complex constants. If the zeros and poles of each $f_j(z)$ lie in a closed convex region K, and if $\mu \leqq \arg m_j \leqq \mu+\gamma<\mu+\pi$, then the zeros of $G(z)$ are contained in the region $S(K, \phi)$, which is star-shaped relative to K and consists of all points from which K subtends an angle not less than $\phi=(\pi-\gamma)/(p+q)$. Chapter II concludes with a discussion of the zeros of the polynomials of Van Vleck and of Stieltjes that arise in a generalized Lamé differential equation.

Given a polynomial $f(z)$ of degree n, there is introduced in Chapter III the Laguerre-originated function

$$f_1(z) = nf(z) + (\zeta - z)f'(z),$$

called by the author the *polar derivative* of $f(z)$ relative to the *pole* ζ. It has the property, not shared by $f'(z)$, that under an arbitrary nonsingular linear fractional transformation, those zeros of $f_1(z)$ that are not multiple zeros of $f(z)$ and are not equal to ζ transform into zeros of the transformed function. For $f_1(z)$ there is an invariant form of the Lucas theorem, due to Laguerre: If all the zeros of $f(z)$ lie in a circular region C, and if Z is a zero of $f_1(z)$, then Z and ζ cannot both lie outside of C; and if, moreover, $f(Z)\neq 0$, any circle that passes through Z and ζ either passes through all the zeros of $f(z)$ or it separates these zeros. The Laguerre theorem is a very useful one for subsequent results. By recurrence, polar derivatives of higher order are defined at the end of the chapter.

Chapter IV deals with composite polynomials. First *apolar* polynomials are treated. Two nth degree polynomials

$$f(z) = \sum_{k=0}^{n} C(n, k)A_k z^k, \qquad g(z) = \sum_{k=0}^{n} C(n, k)B_k z^k$$

are apolar if $\sum_{k=0}^{n}(-1)^k C(n, k)A_k B_{n-k}=0$. To a given $f(z)$ there correspond infinitely many apolar polynomials $g(z)$, and Szegö has given a useful test for apolarity: if $L[f(t)]\equiv\sum_{k=0}^{n} l_k a_k=0$, where $a_k = C(n, k)A_k$, then $g(z)=L[(t-z)^n]$ is apolar to $f(z)$. For apolar polynomials there is the theorem of Grace: If f, g are apolar and the zeros of f lie in a circular region C, then at least one zero of g is in C. And from this follows a result of Walsh (that has beautiful application in subsequent portions of the book): Let $\Phi(z_1, \cdots, z_n)$ be a linear symmetric function of its n variables; if C is a circular region containing the points $z_1^{(0)}, \cdots, z_n^{(0)}$ then there exists at least one point ζ in C for which $\Phi(\zeta, \zeta, \cdots, \zeta)=\Phi(z_1^{(0)}, \cdots, z_n^{(0)})$. Applications of these results are then made to the problem of finding regions in which the zeros of certain polynomials lie, when these polynomials are related to others whose zeros lie in specified regions.

Chapter V opens with a two-circle theorem of Walsh that generalizes the Lucas theorem. It deals with the zeros of $(f_1f_2)'$. A corresponding result holds for $(f_1/f_2)'$; and the general case (due to Marden) where f_1f_2, f_1/f_2 are replaced by $f=(f_0f_1\cdots f_q)/(f_{q+1}\cdots f_p)$ is then discussed.

The results of the preceding chapters depend on knowledge of the location of (that is, knowledge of some region containing) all the zeros of a polynomial $f(z)$. Chapter VI now examines the situation when something is known of some, but not of all, of the zeros of $f(z)$. One of the fundamental results, concerning *two* zeros, is the Grace-Heawood theorem: If $f(z_1)=f(z_2)=0$, then at least one zero of $f'(z)$ is contained in the circle of center $(z_1+z_2)/2$ and radius $(1/2)|z_1-z_2|\cot(\pi/n)$, where n is the degree of f. The radius is "best possible." Kakeya proposed the following problem: Given that p zeros of $f(z)$, a polynomial of degree n, are contained in a circle C of radius R, to find the radius of a concentric circle C' that will contain at least $p-1$ zeros of $f'(z)$. He showed that a function $\phi(n, p)$ exists such that $R'=R\phi(n, p)$ will satisfy the problem; but the best value of ϕ is unknown for $p>2$. A result of Marden shows that the choice $\phi(n, p)=\csc(\pi/2q)$, $q=n-p+1$, is permissible. Other results concerning $\phi(n, p)$ are also treated in this chapter.

Up to now the emphasis has been essentially on one of the two problems mentioned as being the subject matter of this book. The remainder of the work concerns itself largely with the other problem, namely, to obtain information on the location of the zeros of a polynomial $f(z)$ in terms of its coefficients. Chapter VII gives bounds for the absolute values of the zeros of $f(z)$ from knowledge of the moduli of the coefficients. Many names are associated with these bounds; in particular, one may mention the well known theorem of Pellet. By taking into account the arguments as well as the moduli of the coefficients, certain refinements of Pellet's result are given, and a number of applications are discussed.

Chapter VIII takes up various generalizations of Landau's theorems that every polynomial $a_0+a_1z+a_nz^n$ ($a_1\neq 0$, $n\geqq 2$) has at least one zero in the circle $|z|\leqq 2|a_0/a_1|$, and every polynomial $a_0+a_1z+a_mz^m+a_nz^n$ ($a_1\neq 0, 2\leqq m<n$) at least one zero in $|z|\leqq(17/3)|a_0/a_1|$. These are examples of the general lacunary polynomial

$$f(z) = a_0 + a_1z + \cdots + a_pz^p + \sum_{j=1}^{k} a_{n_j}z^{n_j}$$

$$(a_0a_p \neq 0,\ 1 \leqq p < n_1 < \cdots < n_k).$$

It is a result of Montel that $R=R(a_0, \cdots, a_p, k)$ exists such that the circle $|z|\leqq R$ contains at least p zeros for every such $f(z)$. Various permissible choices of R (though not the "best possible") are found in this chapter.

Chapter IX discusses the problem of determining the number of zeros of a polynomial in a half-plane, the result being expressed in terms of the number of variations in sign of certain determinants involving the coefficients of $f(z)$. Corresponding results are also obtained for the number of zeros in a given sector. In the final chapter an algorithm is given in terms of which it is possible to determine the number of zeros of a polynomial in a given circle.

The above résumé of the contents of this book gives some indication of the wealth of material to be found therein. The work should arouse interest in the theory of the zeros of polynomials, and the author is to be thanked for presenting the opportunity to learn the present state of that theory. If the reviewer may however be permitted to express a mild regret, it is that the author has not explicitly charted the possible future, for prospective workers in the field; at various stages in the work there seems to be a real need for the author to take the reader into his confidence and point the way to desirable topics as yet incompletely (or not at all) explored.

Although not every proof was checked to the last detail, enough checking leads one to conclude that the book is free of gross errors, and only slight oversights and typographical errors are to be reported. On pages 1 and 2 it is required of certain functions that they be analytic interior to a simple closed curve C and continuous on C, whereas the continuity should be interior to and on C. On page 17 the letter K in Fig. (7, 1) seems to refer to the dotted rectangle, in which case the label R should be used rather than K. From the statements of Theorem (8, 1), page 19, and line 3 of page 20, K subtends an angle less than ϕ at ζ. Consequently, the angle marked ϕ in Fig. (8, 1) seems inconsistent, as does the phrase "where, however, K subtends at ζ the angle ϕ" of line 2, page 20. On page 48, last line, the subscripts on z_1, z_2 should be removed.

In closing, it is perhaps not out of place to observe that many valuable contributions to the theory expounded in this book are due to J. L. Walsh and the author.

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