SOME ELEMENTARY TOPOLOGICAL PROPERTIES OF ESSENTIAL MAXIMAL MODEL CONTINUA

W. R. SCOTT[1]

1. **Introduction.** Let T: $z=t(w)$, $w\in D$, be a bounded continuous transformation from a bounded connected open set D in the w-plane into the z-plane. Radó and Reichelderfer [3, p. 263][2] have defined essential maximal model continua $\gamma(z)$ for such transformations. If A is a set in the z-plane, let $E(A)$ be the point set sum of the e.m.m.c. $\gamma(z)$ for $z\in A$. Thus $E(A)$ is a subset of D. We are primarily interested in the topological properties of $E(A)$. Since most of the reasoning is valid for more general spaces, it is carried out first for these spaces and then in §6 the results are applied to the above situation. These results are also applicable to the set $E^+(A)$ and $E^-(A)$ defined by Reichelderfer in [4].

2. **Preliminaries.** Let T: $y=t(x)$, $x\in X$, be a single-valued continuous transformation from a normal Hausdorff space X into a topological space Y (these terms are used in the sense of [1, chap. 1]). Suppose that for each set $A\subset Y$ there has been defined a set $E(A)\subset X$ such that

$$E(A) = \sum_{y\in A} E(y), \tag{1}$$

$$E(y) = \sum_{\gamma\in S(y)} \gamma, \tag{2}$$

where $S(y)$ is a class of disjoint nonempty closed connected sets $\gamma\subset X$ such that $\gamma\subset T^{-1}(y)$ and having the following property:

P. If $\gamma_0\in S(y_0)$ and if G is any open set such that $\gamma_0\subset G$, then there exists a neighborhood $N(y_0)$ such that if $y\in N(y_0)$ then there exists a $\gamma\in S(y)$ such that $\gamma\subset G$.

Let $f(y)$ be the number of $\gamma\in S(y)$. Thus $f(y)=0, 1, 2, \cdots$, or $+\infty$, no distinction being made between infinite cardinals.

3. **Closed sets A.**

THEOREM 1. *Let T, A, $E(A)$, and $f(y)$ be given as in §2. If A is closed and if $f(y)$, $y\in A$, is upper semi-continuous and bounded on A, then $E(A)$ is closed.*

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PROOF. Deny the theorem. Then there exists a point $x_0 \in C(E(A)) - E(A)$, where $C(E(A))$ is the closure of $E(A)$. Since A is closed and since $E(A) \subset T^{-1}(A)$ which is closed, we have $t(x_0) = y_0 \in A$. Let $f(y_0) = k$. Then there exist disjoint $\gamma_i \in S(y_0)$, $i = 1, \cdots, k$, and $x_0 \cdot \sum \gamma_i = 0$. Since each γ_i is closed, and since X is a normal Hausdorff space, there exist disjoint open sets $G_0, \cdots, G_k$, such that $x_0 \in G_0$, $\gamma_i \subset G_i$, $i = 1, \cdots, k$. By property P, there exist neighborhoods $N_i(y_0)$ such that if $y \in N_i(y_0)$ then there exists a $\gamma \in S(y)$ such that $\gamma \subset G_i$. Since $f(y)$ is upper semi-continuous on A, there is a neighborhood $N'(y_0)$ such that if $y \in N'(y_0) \cdot A$, then $f(y) \leqq k$. Let

$$N(y_0) = N'(y_0) \cdot N_1(y_0) \cdots N_k(y_0).$$

Then if $y \in N(y_0) \cdot A$, all of the $\gamma \in S(y)$ lie in $\sum_{i=1}^{k} G_i$. But since $x_0 \in G_0$ and $G_0 \cdot \sum_{i=1}^{k} G_i = 0$, this implies that $x_0 \notin C(E(A))$. This is a contradiction. Thus $E(A)$ is closed.

It will be noted that the fact that each γ is connected was not used in the proof of this theorem.

4. Connected sets A.

LEMMA 1. *Let T, A, and $E(A)$ be given as in §2. Let $E(A) = E_1 + E_2$ where $E_1 \cdot C(E_2) + C(E_1) \cdot E_2 = 0$. Let $f_i(y)$, $y \in A$, be the number of $\gamma \in S(y)$ such that $\gamma \subset E_i$. Then $f_i(y)$ is lower semi-continuous on A.*

PROOF. Note that since a $\gamma_0 \in S(y_0)$ is connected, either $\gamma_0 \subset E_i$ or or $\gamma_0 \cdot E_i = 0$. Hence $f_1(y) + f_2(y) = f(y)$, $y \in A$. It will be sufficient to show that $f_1(y)$ is lower semi-continuous on A. Let $f_1(y_0) \geqq n$, $y_0 \in A$. If $n = 0$, then $f_1(y) \geqq n$, $y \in A$. If $n > 0$, there exist $\gamma_j \subset E_1 \cdot E(y_0)$, $j = 1, \cdots, n$. Hence $\gamma_1, \cdots, \gamma_n$, and $C(E_2)$ are disjoint closed sets. Therefore there exist disjoint open sets $G_1, \cdots, G_n$ such that $C(E_2) \cdot \sum G_j = 0$ and $\gamma_j \subset G_j$. By property P there exists a neighborhood $N(y_0)$ such that if $y \in N(y_0)$ then there exists a $\gamma_j \subset G_j \cdot E(y)$. If, in addition, $y \in A$, then $f_1(y) \geqq n$. But this proves that $f_1(y)$ is lower semi-continuous on A.

In particular, if $E_2 = 0$, we have that $f(y)$, $y \in A$, is lower semi-continuous on A. Thus, if $A = Y$, this shows that $f(y)$ is lower semi-continuous on Y.

LEMMA 2. *Let A be a connected subspace of Y. Let $g_i(y)$, $y \in A$, be a (finite) integral-valued, lower semi-continuous function, $i = 1, \cdots, n$. Let $g(y) = \sum g_i(y)$, $y \in A$, be upper semi-continuous. Then $g_i(y)$ is constant on A for each i.*

PROOF. The function $g^*(y) = \sum_{j \neq i} g_j(y)$ is lower semi-continuous.

Hence $g_i(y)$ is upper semi-continuous, being the difference of an upper semi-continuous function $g(y)$ and a lower semi-continuous function $g^*(y)$. Since $g_i(y)$ is also lower semi-continuous, it is continuous. Since A is connected, $g_i(y)$ is constant on A.

THEOREM 2. *Let T, A, $E(A)$, and $f(y)$ be given as in §2. Let A be connected, and suppose that $f(y)$ is upper semi-continuous and $f(y) \leqq k < +\infty$, $y \in A$. Then $E(A)$ has at most k components. Let these components be $E_1, \cdots, E_r$, $r \leqq k$. Let $f_i(y)$, $y \in A$, be the number of $\gamma \in S(y)$ such that $\gamma \subset E_i$. Then $f_i(y)$ is constant for each i.*

PROOF. If $E(A)=0$, the theorem is trivial. If $E(A) \neq 0$, let $E_1, \cdots, E_r$ be such that $E_i \neq 0$, $E_i \cdot C(E_j)=0$ if $i \neq j$, and $\sum E_i = E(A)$. Let $f_i(y)$, $y \in A$, be defined as in the theorem. Clearly $f_i(y)$ is integral-valued and $\not\equiv 0$. Moreover, by Lemma 1, $f_i(y)$ is lower semi-continuous on A. Since $f(y) = \sum f_i(y)$ is upper semi-continuous and bounded on A, Lemma 2 is applicable, and $f_i(y) \equiv k_i \geqq 1$, $i=1, \cdots, r$. But since

$$k \geqq f(y) = \sum_{i=1}^{r} f_i(y) \geqq \sum_{i=1}^{r} 1 = r,$$

we have $r \leqq k$. But if $E(A)$ had more than k components, then there would exist $E_1, \cdots, E_{k+1}$ such that $E_i \neq 0$, $E_i \cdot C(E_j)=0$ if $i \neq j$, and $\sum E_i = E(A)$. Hence $E(A)$ has at most k components and the theorem is true.

5. Open connected sets A.

THEOREM 3. *Let T: $y=t(x)$, $x \in X$, be a continuous transformation from a connected, locally connected, locally compact, separable metric space X into a topological space Y. Let A be a connected open set in Y, let $E(A)$ be given as in §2, and suppose that $f(y) \not\equiv 0$ is upper semi-continuous and bounded on A. Let $y_1 \in A$, $y_2 \in A$. Then there exist points $x_1 \in E(y_1)$, $x_2 \in E(y_2)$, and an arc C from x_1 to x_2 such that $T(C) \subset A$.*

PROOF. Let $E_1(A)$ be any component of $E(A)$. Let $f_1(y)$, $y \in A$, be the number of $\gamma \in S(y)$ such that $\gamma \subset E_1(A)$. Then by Theorem 2, $f_1(y) \equiv k_1 > 0$, $y \in A$. Hence there exist an $x_1 \in E(y_1) \cdot E_1(A)$ and an $x_2 \in E(y_2) \cdot E_1(A)$. Let G be the component of $T^{-1}(A)$ which contains $E_1(A)$. Then [2, Theorem I, 14.1] G is open and connected. Hence [**2**, Theorems I, 14.3 and II, 5.2] G is arcwise connected. Therefore there exists an arc $C \subset G$ joining x_1 and x_2. Clearly $T(C) \subset A$.

6. Plane transformations. Let T: $z=t(w)$, $w \in D$, be a bounded continuous transformation from a bounded connected open set D in the w-plane into the z-plane. Radó and Reichelderfer [**3**, p. 263] have defined essential maximal model continua (e.m.m.c.) γ for such transformations. Let $S(z)$ be the class of (disjoint) e.m.m.c. of z. If $\gamma \in S(z)$ then $\gamma \subset T^{-1}(z)$ and $S(z)$ has property P [**3**, Lemmas 2.8 and 2.16]. Define $E(A)$ by equations (1) and (2). The function $f(z)$ becomes the essential multiplicity function $\kappa(z)$ [**3**, p. 263]. The conditions in Theorems 1, 2, and 3 are satisfied, and we have the following result.

COROLLARY. *Let T: $z=t(w)$, $w\in D$, be a bounded continuous transformation from a bounded connected open set D in the w-plane into the z-plane. Let κ (z) be upper semi-continuous on A with $\kappa(z) \leqq k < +\infty$, $z\in A$. Then if A is closed, $E(A)$ is closed; if A is connected, $E(A)$ has at most k components.*

Theorem 3 of course yields a corollary also.

Reichelderfer[3] [**4**] has recently defined a class $S^+(z)$ for such plane transformations as follows. $S^+(z_0)$ consists of all e.m.m.c. γ_0 of z_0 such that if G is any open set containing γ_0 then there exists a finitely-connected Jordan region $R\subset G$ containing γ_0 in its interior and such that $\mu(z_0, T, R)>0$ (see [**3**, p. 263] for notation). Reichelderfer [**4**] shows that $S^+(z)$ has property P. Define $E^+(A)$ by equations (1) and (2). The function $f(z)$ is then denoted by $\kappa^+(z)$. Then Theorems 1, 2, and 3 apply in this case.

Similar definitions and conclusions hold for $E^-(A)$ (see [**4**]).

7. **Examples.** The discussion will be restricted to bounded continuous plane transformations T: $z=t(w)$, $w\in D$, where D is an open simply-connected set, and to the functions $\kappa(z)$ and $E(z)$ mentioned in §6. The following examples shed further light on Theorems 1, 2, and 3. We have let $w=u+iv$, $z=x+iy$, and have used whichever notation is more convenient.

I. In Theorem 1, if $f(z)$ is merely assumed to be bounded, then $E(A)$ is not necessarily closed. For let D be the disc $|w|<2$, and let

$$z = \begin{cases} w, & |w| \leqq 1 \\ w/|w|, & 1<|w|<2. \end{cases}$$

Let A be the closed disc $|z| \leqq 1$. Then $\kappa(z)\leqq 1$, $z\in A$, and $E(A)$ is the set $|w|<1$ which is not closed in D.

II. In Theorem 1, if $\kappa(z)\equiv +\infty$, $z\in A$, then $E(A)$ is not necessarily closed. Sierpinski [**5**] has constructed a continuous function $x=g^*(u)$, $0\leqq u\leqq 1$, such that $\kappa(x, g^*)=+\infty$ for $0<x<1$ except at a countable

[3] The author had the privilege of reading this paper in manuscript.

set of points. A slight modification of his function yields a continuous function $x=g(u)$, $0\leqq u\leqq 1$, such that $\kappa(x, g)\equiv +\infty$, $0<x<1$, and such that $g(1)=1$. To construct the required example, let D be the open rectangle: $0<u<3$, $0<v<1$. Let T be defined by the relations

$$x = \begin{cases} g(u), & 0 < u \leqq 1, \\ (3-u)/2, & 1 < u \leqq 2, \\ 1/2, & 2 < u < 3, \end{cases}$$

$$y = v, \qquad 0 < v < 1.$$

Let A be the line segment: $1/2\leqq x\leqq 3/4$, $y=1/2$. Then $\kappa(z)\equiv +\infty$, $z\in A$, and $E(A)$ is not closed because it contains the segment $3/2<u<2$, $v=1/2$, but not the point $u=2$, $v=1/2$.

III. In Theorem 1, if A is open and $\kappa(z)\equiv k<+\infty$, $z\in A$, it does not follow that $E(A)$ is open. For let D be the disc $|w|<2$, and let

$$z = t(w) = \begin{cases} (w-1)w, & 1 \leqq |w| < 2, \\ |w| - 1, & 0 \leqq |w| < 1. \end{cases}$$

Let A be the disc $|z|<2$. Then it follows that $\kappa(z)\equiv 1$, $z\in A$, and $E(A)$ is the set of points w such that $1\leqq |w|<2$. Hence $E(A)$ is not open.

IV. In Theorem 2, $E(A)$ may be connected, thus having only 1 component, even though $\kappa(z)\equiv k<+\infty$, $z\in A$. For let $z=w^k$, $|w|<2$. Let A be the unit circle $|z|=1$. Then $\kappa(z)\equiv k$, $z\in A$. But $E(A)$ is the unit circle $|w|=1$, and is therefore connected.

V. Theorem 3 does not hold for line segments A. For let D be the open rectangle: $-2<u<2$, $-2<v<2$. Let S be the point set consisting of the line segment $u=0$, $-1\leqq v\leqq 1$, and the curve $v=\sin 2\pi/u$, $0<u\leqq 1$. Let S' be the point set consisting of S and the two line segments $-2<u<0$, $v=0$, and $1<u<2$, $v=0$. Let $d((u, v), S')$ be the distance from the point (u, v) to the set S'. Define the transformation T as follows:

$$x = \begin{cases} u, & -2 < u < 0, \\ 0, & 0 \leqq u \leqq 1, \\ u-1, & 1 < u < 2, \end{cases}$$

$$y = \begin{cases} d((u, v), S'), & (u, v) \text{ above } S', \\ 0, & (u, v) \in S', \\ -d((u, v), S') & (u, v) \text{ below } S'. \end{cases}$$

T is clearly continuous. Let A be the line segment: $-1\leqq x\leqq 1/2$, $y=0$. Then $\kappa(z)\equiv 1$, $z\in A$, and $T^{-1}(A)=E(A)$ consists of the line segments $-1 \leqq u < 0$, $v = 0$, and $1 < u \leqq 3/2$, $v = 0$, and the set S. Hence if $z_1 = (-1, 0)$ and $z_2 = (1/2, 0)$ there is no arc $w_1w_2 \subset D$ such that $t(w_1) = z_1$, $t(w_2) = z_2$, $T(\text{arc } w_1w_2) \subset A$.

VI. There remains the question as to whether, in the case of plane transformations, we could require $C \subset E(A)$ in Theorem 3. This would be impossible in the general case, as may be shown by an example similar to the one described in V.

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Ohio State University