AN INEQUALITY RELATED TO THE ISOPERIMETRIC INEQUALITY

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In this note we shall prove the following theorem.

THEOREM 1. *Let m be the measure of an open subset O of Euclidean n-space, and let $m_1, \cdots, m_n$ be the $(n-1)$-dimensional measures of the projections of O on the coordinate hyperplanes. Then*

$$m^{n-1} \leqq m_1 m_2 \cdots m_n. \tag{1}$$

Note that for n-dimensional intervals with faces parallel to the coordinate hyperplanes, (1) holds with the equality sign.

With any reasonable definition of the $(n-1)$-dimensional measure s of the boundary of O, $s \geqq 2m_i$ for each i, so that (1) gives

$$m^{n-1} \leqq s^n/2^n; \tag{2}$$

this is the isoperimetric inequality, without the best constant. Since the proof of the isoperimetric inequality with the best constant is difficult,[1] and since its applications do not necessarily require the best constant, our elementary proof of the theorem may be of some interest.

We first reduce the problem to a combinatorial one, in the following theorem.

THEOREM 2. *Let S be a set of cubes from a cubical subdivision of n-space; let S_i be the set of $(n-1)$-cubes obtained by projecting the cubes of S onto the ith coordinate hyperplane. Let N and N_i be the numbers of cubes in S and S_i respectively. Then*

$$N^{n-1} \leqq N_1 N_2 \cdots N_n. \tag{3}$$

Assuming Theorem 2, we prove Theorem 1 as follows. Given $\epsilon > 0$, choose a cubical subdivision of n-space into cubes of side δ, with δ so small that if S is the set of cubes interior to O forming the set $\overline{S}$, $\mu(O-\overline{S}) < \epsilon$ ($\mu =$ measure). Then

$$[\mu(\overline{S})]^{n-1} = N^{n-1}\delta^{n(n-1)} \leqq (N_1\delta^{n-1}) \cdots (N_n\delta^{n-1}) \leqq m_1 \cdots m_n,$$

and since ϵ is arbitrary, (1) follows.

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[1] See E. Schmidt, *Über das isoperimetrische Problem in Raum von n Dimensionen*, Math. Zeit. vol. 44 (1939) pp. 689–788.

PROOF OF THEOREM 2.[2] If $n=2$, the theorem is clear; we shall use induction on n. Each cube of S projects into an interval on the first coordinate axis; let $I_1, \cdots, I_k$ be the intervals thus obtained. Let T_i be the set of cubes projecting onto I_i, and let T_{ij} be the set of $(n-1)$-cubes obtained by projecting the cubes of T_i into the jth coordinate hyperplane $(j=2, \cdots, n)$. Let a_i and a_{ij} be the numbers of cubes in T_i and T_{ij} respectively. Clearly

$$\sum_{i=1}^{k} a_i = N, \qquad a_i \leqq N_1 \ (i = 1, \cdots, k), \tag{4}$$

$$\sum_{i=1}^{k} a_{ij} = N_j \qquad (j = 2, \cdots, n). \tag{5}$$

Also, by induction,

$$a_i^{n-2} \leqq a_{i2} \cdots a_{in} \qquad (i = 1, \cdots, k). \tag{6}$$

From (6) and the second part of (4) we obtain

$$a_i^{n-1} \leqq N_1 a_{i2} \cdots a_{in} \qquad (i = 1, \cdots, k).$$

Now using successively the first part of (4), the above inequality, Hölder's inequality, and (5), we see that

$$\begin{aligned} N = \sum_{i=1}^{k} a_i &\leqq \sum_{i=1}^{k} N_1^{1/(n-1)} \prod_{j=2}^{n} a_{ij}^{1/(n-1)} \\ &\leqq N_1^{1/(n-1)} \prod_{j=2}^{k} \left(\sum_{i=1}^{m} a_{ij} \right)^{1/(n-1)} = \prod_{j=1}^{n} N_j^{1/(n-1)}, \end{aligned}$$

as required.

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[2] The authors are indebted to M. R. Demers for a simplification in the proof of this theorem.