ON THE FREQUENCY OF PAIRS OF SQUARE-FREE NUMBERS WITH A GIVEN DIFFERENCE

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If k is a positive integer, then the function

$$f(x) = f(x, k) = \sum_{n \leqq x} | \mu(n)\mu(n + k) |$$

enumerates the number of pairs of square-free integers with fixed difference k such that the smaller of the two does not exceed x. The purpose of the present note is to establish the following result.

THEOREM. *As* $x \to \infty$ *we have*

$$f(x) = \prod_p \left(1 - \frac{2}{p^2}\right) \prod_{p^2 | k} \left(1 + \frac{1}{p^2 - 2}\right) x + O(x^{2/3} \log^{4/3} x),$$

where the O-constant may depend upon k.

In a previous publication[1] I considered the more general sum

$$F(x) = \sum_{n \leqq x} \mu_r(n + k_1) \cdots \mu_r(n + k_s),$$

where $k_1, \cdots, k_s$ are distinct integers, r is an integer greater than 1, and $\mu_r(n)$ is defined as 0 or 1 according as n is or is not divisible by the rth power of a prime. I showed that, for $x \to \infty$,

$$F(x) = Ax + O(x^{2/(r+1)+\epsilon}), \tag{1}$$

where A is a constant which can be expressed as an infinite series or else as a product ranging over primes. The asymptotic formula (1) generalized and sharpened an earlier estimate due to Pillai.[2] The present note furnishes a slight improvement on (1) for the case $r = 2$, $s = 2$. The factor x^ϵ in (1) arose from the expression $\max_{\nu \leqq x} d(\nu)$, and could not, therefore, be replaced by a power of $\log x$ by the method previously used.

Our notation is as follows. The letters x, y denote positive numbers; all other small letters denote positive integers unless otherwise stated, and p is reserved for primes.

The O-notation refers to the passage $x \to \infty$, and O-constants de-

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[1] L. Mirsky, *Note on an asymptotic formula connected with r-free integers*, Quart. J. Math. Oxford Ser. vol. 18 (1947) pp. 178–182.

[2] S. S. Pillai, *On sets of square-free integers*, J. Indian Math. Soc. N.S. vol. 2 (1936) pp. 116–118.

pend at most upon k.

(a, b) denotes the highest common factor of a and b.

$d(n)$ denotes the number of divisors of n.

We write

$$N_1(x, a, b, k) = \sum_{au-bv=k,\, bv\leqq x} 1; \qquad N_2(x, a, b, k) = \sum_{au^2-bv^2=k,\, bv^2\leqq x} 1.$$

We begin with two preliminary estimates.

LEMMA 1. *If* $(a, b)\nmid k$, *then* $N_1(x, a, b, k)=0$.
If $(a, b)\mid k$, *then* $N_1(x, a, b, k)=x(a, b)/ab+O(1)$.

This result is effectively case $s=2$ of Lemma 2 of my paper referred to above. The proof is extremely simple and may be left to the reader.

LEMMA 2. $N_2(x, a, b, k)=O(\log x)$.

For the case when ab is not a square it was shown by Estermann[3] that

$$N_2(x, a, b, k) \leqq 2d(k)\{\log (x + k) + 1\}.$$

If, on the other hand, ab is a square, the required result follows trivially. For write $a=l^2t$, $b=m^2t$. If $t\nmid k$, then $N_2(x, a, b , k)=0$, while if $t\mid k$, then

$$\begin{aligned} N_2(x, a, b, k) &= N_2(xt^{-1}, l^2, m^2, kt^{-1}) \\ &\leqq \sum_{l^2u^2-m^2v^2=kt^{-1}} 1 \leqq \sum_{r^2-s^2=kt^{-1}} 1 = O(1). \end{aligned}$$

We now come to the proof of the theorem. Since

$$|\mu(n)| = \sum_{m^2\mid n} \mu(m)$$

we have

$$f(x) = \sum_{a^2c-b^2d=k,\, b^2d\leqq x} \mu(a)\mu(b) = \sum\nolimits_1 + \sum\nolimits_2, \tag{2}$$

say, where $ab\leqq y$ in $\sum_1$ and $ab>y$ in $\sum_2$. Here y denotes a function of x to be fixed later.

Writing

$$K = \sum_{(a,b)^2\mid k} \mu(a)\mu(b)\,\frac{(a, b)^2}{a^2b^2}$$

[3] T. Estermann, *Einige Sätze über quadratfreie Zahlen*, Math. Ann. vol. 105 (1931) pp. 653–662, Hilfssatz 2 (Anhang).

we have, by Lemma 1,

$$\sum\nolimits_1 = \sum_{ab \leqq y} \mu(a)\mu(b) N_1(x, a^2, b^2, k)$$

$$= x \sum_{ab \leqq y, (a^2, b^2) | k} \mu(a)\mu(b) \frac{(a, b)^2}{a^2 b^2} + O(y \log y)$$

$$= Kx + O\left(x \sum_{ab > y} \frac{1}{a^2 b^2} \right) + O(y \log y)$$

$$= Kx + O(xy^{-1} \log y) + O(y \log y). \tag{3}$$

Again, by Lemma 2,

$$\left| \sum\nolimits_2 \right| \leqq \sum_{a^2 c - b^2 d = k, b^2 d \leqq x, ab > y} 1 \leqq \sum_{a^2 c - b^2 d = k, b^2 d \leqq x, cd < x(x+k)y^{-2}} 1$$

$$= \sum_{cd < x(x+k)y^{-2}} N_2(x, c, d, k) = O(x^2 y^{-2} \log^2 x). \tag{4}$$

Putting $y = x^{2/3} \log^{1/3} x$ we obtain, by (2), (3), and (4),

$$f(x) = Kx + O(x^{2/3} \log^{4/3} x).$$

Finally, it is clear that $K = \prod_p \chi_p$, where

$$\chi_p = \sum_{u, v = 0, 1; (p^u, p^v)^2 | k} \mu(p^u)\mu(p^v) \frac{(p^u, p^v)^2}{p^{2u+2v}} = \begin{cases} 1 - 1/p^2 & \text{if } p^2 \mid k, \\ 1 - 2/p^2 & \text{if } p^2 \nmid k. \end{cases}$$

The theorem now follows at once.

It may be worth mentioning that the method of the present note also enables us to investigate the sums

$$\sum_{n \leqq x} \left| \mu(q_1 n + k_1) \mu(q_2 n + k_2) \right|,$$

and

$$\sum_{n \leqq x} \left| \mu((q_1 n + k_1)(q_2 n + k_2)) \right|$$

(where k_1, k_2, q_1, q_2 are given integers), and to obtain asymptotic formulae for these sums with error terms of the form $O(x^{2/3} \log^{4/3} x)$.

Another result which can be established by an argument analogous to that used above is as follows.

Let $t_1(n) = O(\log^{\alpha_1} n)$, $t_2(n) = O(\log^{\alpha_2} n)$, where α_1, α_2 are any given real numbers, and let

$$T_1(n) = \sum_{m^2 | n} t_1(m), \qquad T_2(n) = \sum_{m^2 | n} t_2(m).$$

Then

$$\sum_{n \leqq x} T_1(n) T_2(n+k) = Cx + O(x^{2/3} \log^{\alpha_1+\alpha_2+4/3} x),$$

where

$$C = \sum_{(a,b)^2 \mid k} t_1(a) t_2(b) \frac{(a, b)^2}{a^2 b^2} \cdot$$

The theorem we have proved in detail is the special case $t_1(n) = t_2(n) = \mu(n)$ of this result.

Added in proof (September 1949). In a recent paper (Quart. J. Math. Oxford Ser. vol. 20 (1949) pp. 65–79) F. V. Atkinson and Lord Cherwell obtained a generalization of Lemma 2 in which squares are replaced by rth powers. This new estimate enables me to extend the result of the present paper to numbers not divisible by rth powers, and to prove that, for any $r \geqq 2$,

$$\sum_{n \leqq x} \mu_r(n) \mu_r(n+k) = \prod_p \left(1 - \frac{2}{p^r}\right) \prod_{p^r \mid k} \left(\frac{p^r - 1}{p^r - 2}\right) x + O(x^{2/(r+1)} (\log x)^{(r+2)/(r+1)}).$$

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