CYCLIC INVARIANCE UNDER MULTI-VALUED MAPS[1]

A. D. WALLACE

In what follows it is always assumed that X, Y are compact (=bicompact) connected Hausdorff spaces each containing more than one point.

Let f denote a function which assigns to each x in X a *subset* $f(x)$ of Y. We suppose that the sets $\{f(x)\}$ cover Y. By definition

$$f^{-1}(y) = \{x \mid x \in f(y)\}.$$

It is assumed that the sets $\{f^{-1}(y)\}$ cover X. The functions f and f^{-1} play dual roles inasmuch as $f=(f^{-1})^{-1}$. If f is single-valued, then f^{-1} is the inverse of f in the usual meaning of the term. For $A \subset X$, $B \subset Y$ we define

$$f(A) = \cup\{f(x) \mid x \in A\}, \qquad f^{-1}(B) = \cup\{f^{-1}(y) \mid y \in B\}.$$

When f is single-valued we know that continuity is equivalent to the assertion that A, B closed imply $f(A)$, $f^{-1}(B)$ closed. When f is multi-valued we take this as a *definition of continuity*. It does not follow, as in the single-valued case, that $f^{-1}(B)$ is open if B is open. These definitions include both a single-valued map (=continuous function) and its inverse.

In this note we show that certain theorems of analytic topology carry over to multi-valued maps (=continuous multi-valued functions as defined above). Some of our results are new even for single-valued maps. Except for fixed-point theorems there seem to be no results in the literature for multi-valued maps.

We say that f is *anarthric* if it is continuous and if for $y \in Y$ no $x \in X - f^{-1}(y)$ separates $f^{-1}(y)$ in X. If f is single-valued and non-alternating, then f is anarthric. See Wallace [2],[2] [3], and [4] and Whyburn [5] and [6]. It is clear that if f is the inverse of a single-valued map, then f is anarthric.

For simplicity we write $P \mid Q$ to mean that the sets P and Q are mutually separated. Also if p, $q \in X$, then $p \sim q$ means that no point separates p and q in X.

THEOREM 1. *In order that the multi-valued map f be anarthric each of the following conditions is both necessary and sufficient:*

Received by the editors June 28, 1948.

[1] This work was done under Contract N7-onr-434, Task Order 3, Navy Department, The Office of Naval Research.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

(i) *If* $X=M\cup N$, *where* M *and* N *are continua meeting in a cutpoint* x, *and* K *is any continuum meeting* M, *then* $f(M\cap K)=f(M)\cap f(K)$.

(ii) *If* H *is a subcontinuum of* Y *and* K *is a subcontinuum of* X *and* $K\cap f^{-1}(H)=P\cup Q$, $P|Q$, *then there exist points* $p\in P$, $q\in Q$ *such that* $p\sim q$.

PROOF. We show that (i) holds if f is anarthric. Now the conclusion follows at once if K is disjoint with either $M-x$ or $N-x$. We assume therefore that K meets both of these sets and that $y\in f(K)\cap f(M)-f(K\cap M)$, the inclusion $f(M\cap K)\subset f(K)\cap f(M)$ clearly holding. Then $f^{-1}(y)$ meets both K and M but not $K\cap M$. Hence x is not in $f^{-1}(y)$ since $x\in K\cap M$. But $f^{-1}(y)$ meets both $M-x$ and $N-x$, a contradiction.

Next, (i) implies (ii). For let K_0 be a continuum contained in K and irreducible between the disjoint closed sets P and Q. Let $p\in P\cap K_0$, $q\in Q\cap K_0$ and suppose that x separates p and q in X. Then $p\cup q\cup(K_0-(P\cup Q))$ is connected and so contains x. Thus x is not in $f^{-1}(H)$ and so $f(x)$ does not intersect H. We have a decomposition $X=M\cup N$ with M and N closed, $M\cap N=x$ and $p\in M$, $q\in N$. Now $f(x)=f(M)\cap f(N)$ as we see by taking the K of (i) to be the present N. But $H=(H\cap f(M))\cup(H\cap f(N))$ and so $H\cap f(M)\cap f(N)$ is not void since H is a continuum contrary to the fact that this intersection is $H\cap f(x)$.

Finally (ii) implies that f is anarthric by taking $H=y$ and $K=X$.

We remark that it is *sufficient* to take $K=N$ in (i) and in (ii) to take $K=X$.

We recall briefly some definitions and results mostly contained in Wallace [2]. These reduce to the well known cyclic element theory if X, Y are metric and locally connected. See Whyburn [5].

By an A-set we mean a closed set H such that if $z\in X-H$, then $X=M\cup N$ with $M\cap N=x$, $(M-x)|(N-x)$, $H\subset M$, $z\in N-x$. It is easily seen that an A-set is a chain (Wallace [2]) and hence a continuum, that the intersection of any collection of A-sets is again an A-set and that the union of two intersecting A-sets is also an A-set.

A prime-chain is a chain which is either an end point, a cutpoint or a nondegenerate minimal chain. One can replace "chain" by "A-set" in the last sentence. It is readily seen that if a chain A is met by a prime-chain E in two points or in a non-cutpoint, then $E\subset A$.

A nodal set is a closed set which meets the closure of its complement in a single point. It is readily seen that an A-set is the intersection of all the nodal sets containing it and (since each nodal set is an A-set) that any intersection of nodal sets is an A-set.

THEOREM 2. *In order that the multi-valued map f be anarthric it is necessary and sufficient that for any A-set H in X and any subcontinuum K of X meeting H we have* $f(H\cap K)=f(H)\cap f(K)$.

PROOF. Suppose that f is anarthric. It is enough to show that, for any y in Y, $f^{-1}(y)\cap H$, $f^{-1}(y)\cap K$ nonvoid imply $f^{-1}(y)\cap H\cap K$ nonvoid. Now $H\cup K$ is a continuum and so by (ii) there exist points $p\in H\cap f^{-1}(y)$, $q\in K\cap f^{-1}(y)$ with $p\sim q$, assuming of course that our implication is not valid. Let E be the prime-chain containing $p\cup q$ (Wallace [2]). Then $E\cap H$ nonvoid implies that $E\cup H$ is an A-set. Thus, since K is a continuum, we know that $K\cap(E\cup K)=(K\cap E)\cup(K\cap H)$ is connected and so $E\cap H\cap K$ is not void. Hence E must contain two distinct points of H since $p\in f^{-1}(y)$ and this latter set does not meet $H\cap K$ by assumption. From this it follows that $E\subset H$ and so $q\in H$, a contradiction.

The sufficiency is readily inferred from the remarks following Theorem 1. The result fails unless it is required that H and K meet. For if X is the union of the unit circle and the segment from (1, 0) to (2, 0) and Y is the unit circle and f is the map X onto Y carrying the segment into (1, 0), then taking $Y=H$ and $K=(2, 0)$ we see that the conclusion fails.

THEOREM 3. *In order that the multi-valued map f be anarthric it is necessary and sufficient that if* $\{A\}$ *is any collection of A-sets with the finite intersection property, then* $f(\bigcap A)=\bigcap f(A)$.

PROOF. If f is anarthric it is sufficient to prove that, if $y\in Y$, the proposition "$f^{-1}(y)$ meets every set in $\{A\}$" implies "$f^{-1}(y)$ meets $\bigcap A$." To this end show that $\{f^{-1}(y)\cap A\}$ has the finite intersection property. Or, for any $A_1, A_2, \cdots, A_n$ in $\{A\}$ we have $f^{-1}(y)\cap A_1\cap\cdots\cap A_n$ nonvoid. Now by Theorem 2 we see that $f(A_1\cap\cdots\cap A_n)=f(A_1)\cap\cdots\cap f(A_n)$. Thus if $f^{-1}(y)$ intersects every A_i, then $f^{-1}(y)$ also intersects $A_1\cap\cdots\cap A_n$.

The sufficiency follows from the fact that, in Theorem 2, it is enough to take K an A-set.

THEOREM 4. *If f is anarthric and the image of each cutpoint is a point, then the image of a nodal set is a nodal set.*

This follows without difficulty from (i) of Theorem 1. The result is false if the condition, that the image of a cutpoint be a point, is deleted. In the (u, v) plane let Y be the circle $u^2+v^2=4$ and X the union of the circles $(u+1)^2+v^2=1$, $(u-1)^2+v^2=1$. Define $g(u, v)=(u, (2u-u^2)^{1/2})$ if v is non-negative and $g(u, v)=(u, -(2u-u^2)^{1/2})$ if v is nonpositive. Let $f=g^{-1}$; then f is anarthric but the left-hand circle is mapped by f into the left-hand semicircle of Y, which is not a nodal set.

THEOREM 5. *If f is anarthric and the image of a cutpoint is a point, then the image of an A-set is an A-set or a point.*

PROOF. If H is an A-set, then H is the intersection of all the nodal sets $\{N\}$ which contain it. By Theorems 3 and 4 we have $f(H) = \bigcap f(N)$. But each $f(N)$ is a nodal set and thus an A-set. Then $f(H)$ is an A-set since it is an intersection of A-sets.

In the case in which f is non-alternating and X is a Peano space this result is due to G. E. Schweigert.

Let us denote, for any non-null set A, the intersection of all the A-sets which contain A by $C(A)$. It then follows from Theorem 5 (see the proof of (3.14) in Wallace [2]) that we have the following corollaries.

COROLLARY. *For any nonempty set $A \subset X$, $C(f(A)) \subset f(C(A))$.*

COROLLARY. *Let f be a single-valued map of X onto Y such that $f^{-1}(y)$ is a point for each cutpoint y in Y. Then the inverse of an A-set is an A-set or a point.*

According to Kelley [1] a central set is an intersection of a finite number of nodal sets. From Theorems 3 and 4 we have the following theorem.

THEOREM 6. *If f is anarthric and the image of each cutpoint is a point, then the image of a central set is a central set.*

THEOREM 7. *In order that a multi-valued map be anarthric it is necessary and sufficient that no A-set separate the inverse of a point.*

PROOF. The condition is clearly sufficient since each cutpoint is an A-set. Suppose that some A-set A separates the inverse of a point so that we have $X - A = U \cup V$, $U | V$, with $f^{-1}(y)$ meeting both U and V but not A. Now $A \cup U$ and $A \cup V$ are A-sets, say H and K. Then $f^{-1}(y)$ intersects both H and K but not $H \cap K$ contrary to the fact $f(H \cap K) = f(H) \cap f(K)$,

Our next result generalizes a noteworthy theorem of G. T. Whyburn [5].

THEOREM 8. *Let f be an anarthric map such that the image of a cutpoint is a point and let E be a prime-chain in Y. Then there is a prime-chain F in X such that $E \subset f(F)$. If F' is any other prime-chain in X, then $f(F')$ meets E in at most one point.*

PROOF. We may suppose that E is nondegenerate. Using the Hausdorff maximality principal (Zorn's lemma) we see that there exists a collection $\{A\}$ of A-sets maximal relative to the properties that (i) $E \subset f(A)$ and (ii) no finite collection of A's has a void intersection. Let F be the intersection of these A's so that, since $f(F) = \bigcap f(A)$, we know that E is a subset of $f(F)$. Now if F is a point it is either an end point or a cutpoint and so a prime-chain. Suppose that F contains more than one point and is not a prime-chain. Then $X = M \cup N$ with $M \cap N = x$, $(M-x) | (N-x)$ with F meeting each of these separands. We conclude that $Y = f(M) \cup f(N)$, $f(x) = y = f(M) \cap f(N)$. Now if E is contained in $f(M)$, then also $E \subset f(M) \cap f(F) = f(M \cap F)$. Since M and F are A-sets so also is $M \cap F$ and this latter is a proper subset of F contrary to the maximality of $\{A\}$. Accordingly we conclude that E meets both $f(M) - y$ and $f(N) - y$, an impossibility since E is a prime-chain.

Let F' be a prime-chain distinct from F such that $f(F')$ contains two points of E. Then $X = M \cup M'$ where M, M' are closed, intersect in a cutpoint x and $F \subset M$, $F' \subset M'$. Then $f(x) = y \in Y$. As before $Y = f(M) \cup f(M')$. If $f(M') \subset f(M)$, then $f(M') = y$ contrary to the fact that $f(F') \subset f(M')$ and contains two points. It then follows that, as above, the point y cuts E in X, a contradiction.

BIBLIOGRAPHY

1. J. L. Kelley, *Simple links and fixed sets under continuous mappings*, Amer. J. Math. vol. 69 (1947) p. 349.

2. A. D. Wallace, *Monotone transformations*, Duke. Math. J. vol. 9 (1942) pp. 487–506.

3. ———, *On non-alternating transformations*, Bull. Amer. Math. Soc. vol. 49 (1943) p. 62.

4. ———, *A structural property of transformations*, Bull. Amer. Math. Soc. vol. 49 (1943) p. 699.

5. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942.

6. ———, *Uniqueness of the inverse of a transformation*, Duke Math. J. vol. 12 (1945) pp. 317–323.

THE TULANE UNIVERSITY