A CONVEX METRIC FOR A LOCALLY CONNECTED CONTINUUM

R. H. BING

A topological space is metrizable if there is a distance function $D(x, y)$ such that if x, y, z are points, then

$$D(x, y) \geqq 0, \text{ the equality holding only if } x = y, \tag{1}$$

$$D(x, y) = D(y, x) \text{ (symmetry)}, \tag{2}$$

$$D(x, y) + D(y, z) \geqq D(x, z) \text{ (triangle condition)}, \tag{3}$$

$$D(x, y) \text{ preserves limit points.} \tag{4}$$

By (4) we mean that x is a limit point of the set T if and only if for each positive number ϵ there is a point of T at a positive distance from x of less than ϵ. We say that the metric $D(x, y)$ is convex if for each pair of points x, y there is a point u such that

$$D(x, u) = D(u, y) = D(x, y)/2. \tag{5}$$

A subset M of a topological space S is said to have a convex metric (even though S may have no metric) if the subspace M of S has a convex metric.

It is known [5][1] that a compact continuum is locally connected if it has a convex metric. The question has been raised [5] as to whether or not a compact locally connected continuum M can be assigned a convex metric. Menger showed [5] that M is convexifiable if it possesses a metric D such that for each point p of M and each positive number ϵ there is an open subset R of M containing p such that each point of R can be joined in M to p by a rectifiable arc of length (under D) less than ϵ. Kuratowski and Whyburn proved [4] that M has a convex metric if each of its cyclic elements does. Beer considered [1] the case where M is one-dimensional. Harrold found [3] M to be convexifiable if it has the additional property of being a plane continuum with only a finite number of complementary domains.

We shall show that if M_1 and M_2 are two intersecting compact continua with convex metrics D_1 and D_2 respectively, then there is a convex metric D_3 on M_1+M_2 that preserves D_1 on M_1 (Theorem 1). Using this result, we show that any compact n-dimensional locally connected continuum has a convex metric (Theorem 6). We do not

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[1] Numbers in brackets refer to the references cited at the end of the paper.

answer the question: *Does each compact locally connected continuum have a convex metric?*[2]

In my paper *Extending a metric* it is shown [**2**, Theorem 5] that if K is a closed subset of the metrizable space S and D_1 is a metric on K, then there is a metric D_2 on S that preserves D_1 on K. The following result is a modification of that result.

THEOREM 1. *If M_1 and M_2 are two intersecting compact continua with convex metrics D_1 and D_2 respectively, there is a convex metric D_3 on M_1+M_2 that preserves D_1 on M_1.*

PROOF. Let $G(x)$ be the least upper bound of $D_1(p, q)$ for all points p, q of $M_1 \cdot M_2$ such that $D_2(p, q) \leqq x$. Then

$$G[D_2(p, q)] \geqq D_1(p, q) \qquad (p,\ q \text{ elements of } M_1 \cdot M_2). \tag{6}$$

Since $G(x)$ is a monotone nondecreasing function of x that approaches zero as x approaches zero, there is a function $F(x)$ $(x>0)$ such that $F(x)$ approaches zero as x approaches zero, $F(x) \geqq G(x)$, and the derivative of $F(x)$ with respect to x is a continuous monotone nonincreasing function greater than one.

If C is an arc which lies except for possibly its end points in $M_2 - M_1 \cdot M_2$ and $\int_C F'[D_2(p, M_1)]ds$ exists, where $F'(x)$ represents the derivative of $F(x)$ with respect to x, p is a variable point of C, s is the length along C under D_2, and $D_2(p, M_1)$ is the greatest lower bound of $D_2(p, q)$ for all points q of M_1, we define the length $L_0(C)$ of C under D_0 to be $L_0(C) = \int_C F'[D_2(p, M_1)]ds$. We note that $L_0(C)$ is not defined for all arcs C. However, if r and q are two points which lie on an arc C with such a length $L_0(C)$, then we call $D_0(r, q)$ the greatest lower bound of $L_0(C)$ for all such arcs C from r to q. Since $F'(x) > 1$, $D_0(r, q) \geqq D_2(p, q)$.

If r and q are two points of M_1 and $D_0(r, q)$ is defined, then

$$D_0(r, q) \geqq D_1(r, q) \tag{7}$$

because for each positive number ϵ there is a curve C from r to q whose interior lies in $M_1 - M_1 \cdot M_2$ such that

$$\begin{aligned} D_0(r, q) + \epsilon > \int_C F'[D_2(p, M_1)]ds \geqq \int_C F'(s)ds \\ = F(\text{length } C \text{ under } D_2) \geqq G[D_2(r, q)] \geqq D_1(r, q). \end{aligned} \tag{8}$$

The first "$\geqq$" relationship in (8) follows from the facts that $F'(x)$ is a

[2] Since this paper was submitted, both E. E. Moise [**8**] and the author [**9**], working independently, have answered this question in the affirmative.

nonincreasing function and $s \geqq D_2(p, M_1)$ since the end points of C lie on M_1; the second "$\geqq$" results from the facts that $F(x) \geqq G(x)$, length C under $D_2 \geqq D_2(r, q)$, and $G(x)$ is a monotone nondecreasing function; the third "$\geqq$" is a consequence of (6).

If q is a point of $M_2 - M_1 \cdot M_2$ and C is a shortest arc from q to M_1 under D_2, then

$$L_0(C) = \int_C F'[D_2(p, M_1)]ds = \int_C F'(s)ds = F[D_2(q, M_1)].$$

Hence, if $p_1, p_2, \cdots$ is a sequence of points of $M_2 - M_1 \cdot M_2$ such that $D_2(p_i, M_1)$ approaches zero as i increases without limit, then $D_0(p_i, M_1)$ approaches zero as i increases without limit.

If p and q are two points of M_1, we define $D_3(p, q)$ to be $D_1(p, q)$; if p is a point of M_1 and q is a point of $M_2 - M_1 \cdot M_2$, we define $D_3(p, q)$ to be the greatest lower bound of $D_1(p, a) + D_0(a, q)$ for all points a of M_1; if both p and q are points of $M_2 - M_1 \cdot M_2$, we define $D_3(p, q)$ to be the minimum of $D_0(p, q)$ and the greatest lower bound of $D_0(p, a) + D_1(a, b) + D_0(b, q)$, where a is a point of M_1 and so is b. If $p = q$, we define $D_3(p, q)$ to be equal to zero.

The above definition of $D_3(p, q)$ is equivalent to defining $D_3(p, q)$ to be the greatest lower bound of the lengths of all arcs C from p to q where length in M_1 is measured under D_1 and length in $M_2 - M_1 \cdot M_2$ is measured under D_0. It follows from (7) that we need only consider those arcs C which intersect M_1 in a connected piece if at all.

Now the function $D_3(p, q)$ may be shown to satisfy conditions (1), (2), (3), (4), and (5). Hence, it is a convex metric for $M_1 + M_2$ that preserves D_1 on M_1.

THEOREM 2. *If D_1 and D_2 are convex metrics for the intersecting compact continua M_1 and M_2 respectively and $D_2 \geqq D_1$ on $M_1 \cdot M_2$, then for each positive number ϵ there is a convex metric D_3 for $M_1 + M_2$ such that $D_3 = D_1$ on M_1, $D_3 \leqq D_2$ on M_2, and the diameter under D_3 of each component of $M_2 - M_1 \cdot M_2$ is less than ϵ plus twice the diameter under D_1 of the boundary with respect to $M_1 + M_2$ of this component.*

PROOF. Define $E(p, q)$ to be the greatest lower bound of all sums of the type $f(p, p_1) + f(p_1, p_2) + \cdots + f(p_n, q)$ where adjacent points of $p, p_1, \cdots, p_n, q$ belong to the same one of the continua M_1, M_2 and $f(p_i, p_j)$ is $D_1(p_i, p_j)$ or $D_2(p_i, p_j)$ according as $p_i + p_j$ is or is not a subset of M_1. Since $D_2 \geqq D_1$ on $M_1 \cdot M_2$, n need be no larger than 2 for $E(p, q)$ to attain this greatest lower bound. The convex metric E on $M_1 + M_2$ preserves D_1 on M_1 and $E \leqq D_2$ on M_2.

Let X be the set of all points p of $M_2 - M_1 \cdot M_2$ such that the distance from p to M_1 under E is greater than one-half the diameter under E of the boundary with respect to M_1+M_2 of the component of $M_2-M_1\cdot M_2$ containing p.

Let n be a positive number so small that ϵ/n is greater than twice the diameter of M_2 under E. If C is a rectifiable (under E) arc in M_1+M_2 from p to q, define $L(C)$ to be the greatest lower bound of all sums of the type $f(p, p_1)+f(p_2, p_3)+ \cdots +f(p_k, q)$ where $p_1, p_2, \cdots, p_k$ are points of C and $f(p_i, p_j)$ is either n times or 1 times the length under E of the subarc of C from p_i to p_j according as this subarc is or is not a subset of X. If $D_3(p, q)$ is defined to be the greatest lower bound of all such values $L(C)$, D_3 satisfies the conditions of Theorem 2.

THEOREM 3. *Suppose M_2 is a compact continuum with a convex metric, M_2 lies in a complete locally connected space S with a metric D, each component of $S-M_2$ is of diameter under D of less than θ and M_1 is a subcontinuum of M_2 with a convex metric D_1 such that*

$$D_1(p, q) \leqq D(p, q) \qquad \text{if } D_1(p, q) > \theta \ (p, q \text{ elements of } M_1). \tag{9}$$

For each positive number ϵ there is a continuum M_3 containing M_2 and a convex metric D_3 for M_3 such that D_3 preserves D_1 on M_1 and the boundary of each component of $S-M_3$ is of diameter less than $\theta+\epsilon$ under D_3.

PROOF. By Theorem 1, there is a convex metric D_2 for M_2 that preserves D_1 on M_1. Let n be an integer so large that

$$nD_2(p, q) > D(p, q) \qquad \text{if } D(p, q) > \epsilon/8 \ (p, q \text{ elements of } M_2). \tag{10}$$

Let X denote the collection of all pairs of points (x, y) such that both x and y are points of the boundary of the same component of $S-M_2$ and

$$nD_2(x, y) > \theta + \epsilon. \tag{11}$$

There is a finite subcollection X' of X such that for each element (x, y) of X there is an element (x', y') of X' such that both x' and y' are accessible from the same component of $S-M_2$ and

$$nD_2(x, x') + nD_2(y', y) < \epsilon/2. \tag{12}$$

Let $C_1, C_2, \cdots, C_j$ be a finite collection of components of $S-M_2$ irreducible with respect to the property that for each element of X' there is an integer i less than or equal to j such that both points of this element of X' are accessible from C_i.

There is a dendron T_i $(i=1, 2, \cdots, j)$ such that T_i lies except for its ends in C_i and the sum of the ends of T_i is a finite subset Y_i of the boundary of C_i such that Y_i contains all points of the sum of the elements of X' that are accessible from C_i and for each point q of the boundary of C_i there is a point r of Y_i such that $nD_2(r, q) < \epsilon/2$. Let $D(T_i)$ be a convex metric for T_i such that if p and q are two end points of T_i, then

$$\theta + \epsilon/4 < D(T_i; p, q) < \theta + \epsilon/2. \tag{13}$$

Let E be a metric for $M_2 + \sum T_i = M_3$ such that the distance between two points of M_3 under E is the greatest lower bound of the lengths of arcs containing them where length is measured by nD_2 in M_2 and by $D(T_i)$ in T_i. If (x, y) is an element of X, it follows from (12) and (13) that

$$E(x, y) < \theta + \epsilon. \tag{14}$$

The diameter of the boundary of each component of $S - M_3$ is less than $\theta + \epsilon$ under E, for suppose p and q are two points of this boundary; if $p + q$ is a subset of M_2, $E(p, q) < \theta + \epsilon$ by (11) and (14); if neither p nor q is a point of M_2, both belong to some T_i whose diameter is less than $\theta + \epsilon/2$; if p is an interior point of a T_i which does not contain q, there is an end point r of T_i such that $nD_2(r, q) < \epsilon/2$ and then $E(p, q) \leqq D(T_i; p, r) + nD_2(r, q) < \theta + \epsilon$.

We shall show that if r and s are two points of M_1, then $D_1(r, s) \leqq E(r, s)$. Suppose this is not the case and that rs is an arc from r to s in M_3 whose length is less than $D_1(r, s)$ under E. If $D_1(r, s) \leqq \theta$, then rs is a subset of M_2 alone and

$$E(r, s) = nD_2(r, s) \geqq D_2(r, s) = D_1(r, s). \tag{15}$$

Suppose rs is not a subset of M_2 and $p_1p_2, p_3p_4, \cdots, p_{2j-1}p_{2j}$ are the subarcs of rs which lie except for their end points in $M_3 - M_2$ where $p_{2i-1}p_{2i}$ precedes $p_{2i+1}p_{2i+2}$ on rs in the order from r to s. Let $Z(t)$ be 0 or t according as t is less than $\epsilon/8$ or not. If $D_1(r, s) > \theta$, it follows from (9), (3), and (10) that

$$\begin{aligned} D_1(r, s) &\leqq D(r, s) \leqq D(r, p_1) + D(p_1, p_2) + \cdots + D(p_{2j}, s) \\ &< D(r, p_1) + \theta + D(p_2, p_3) + \cdots + \theta + D(p_{2j}, s) \\ &< Z[D(r, p_1)] + Z[D(p_2, p_3)] + \cdots + Z[D(p_{2j}, s)] \\ &\quad + j\theta + (j + 1)\epsilon/8 \\ &\leqq nD_2(r, p_1) + nD_2(p_2, p_3) + \cdots + nD_2(p_{2j}, s) \\ &\quad + j(\theta + \epsilon/4) \\ &\leqq \text{length } rs \text{ under } E. \end{aligned} \tag{16}$$

It follows from (15) and (16) that $E(r, s) \geqq D_1(r, s)$.

It follows from Theorem 2 that there is a convex metric D_3 for M_3 such that $D_3 = D_1$ on M_1 and $D_3 \leqq E$. The boundary of each component of $S - M_3$ is of diameter less than $\theta + \epsilon$ under D_3 because it is under E.

THEOREM 4. *Suppose M is a compact locally connected continuum such that if p is a point contained in an open subset R_1 of M, there is an open subset R_2 of R_1 containing p such that the boundary of R_2 with respect to M is a subset of a subcontinuum of M with a convex metric. Then M has a convex metric.*

PROOF. Let F be a metric for M. We shall show that there is a collection of continua $M_1, M_2, \cdots$ in M and a collection of metrics $D_1, D_2, \cdots$ such that:

(a) M_{i+1} contains M_i.

(b) D_i is a convex metric for M_i.

(c) D_{i+1} preserves D_i on M_i.

(d) Under F, each component of $M - M_i$ is of diameter less than $1/4^i$.

(e) Under D_i, the boundary (with respect to M) of each component of $M - M_i$ is of diameter less than $1/4^i$.

(f) Under D_{i+1}, the common part of the M_{i+1} and each component of $M - M_i$ is of diameter less than $3/4^i$.

First, we show that if ϵ is a positive number, there is a subcontinuum W of M with a convex metric such that each component of $M - W$ is of diameter less than ϵ under F. By the Heine-Borel Theorem, we find that there is a finite collection G of subcontinua of M such that each element of G has a convex metric, for each point p of M there is an element g of G such that p belongs either to g or to a component of $M - g$ of diameter under F of less than ϵ, and the sum W of the elements of G is a continuum. Each component of $M - W$ is of diameter less than ϵ under F and it follows from Theorem 1 that W has a convex metric.

Let M_1 be a subcontinuum of M with a convex metric E such that each component of $M - M_1$ is of diameter less than $1/4$ under F. A suitable multiple of E gives a metric D_1 for M_1 which will satisfy conditions (b) and (e).

Let n be an integer so large that

$$nF(p, q) > D_1(p, q) \quad \text{if} \quad D_1(p, q) > 1/17 \quad (p, q \text{ elements of } M_1).$$

There is a continuum W in M containing M_1 such that W has a convex metric and the diameter of each component of $M - W$ is less than

$1/17$ under nF. By Theorem 3, there is a continuum M_2 with a convex metric E such that M_2 contains W, E preserves D_1 on M_1, and the diameter under E of the boundary with respect to M of each component of $M-M_1$ is less than $1/4^2$. Applying Theorem 2 we find that M_2 has a convex metric D_2 such that M_2 and D_2 satisfy conditions (a), (b), (c), (d), (e), and (f).

Similarly, there exist continua $M_3, M_4, \cdots$ and convex metrics $D_3, D_4, \cdots$ satisfying conditions (a), (b), (c), (d), (e), and (f). Let D be a function of the pairs of points of M such that $D(p, q)$ is the lower limit of $D_1(p_1, q_1), D_2(p_2, q_2), \cdots$ where $p_1, p_2, \cdots$ and $q_1, q_2, \cdots$ are sequences of points converging to p and q respectively and p_i+q_i is a subset of M_i. We shall show that D preserves limit points in M.

Let ϵ be a positive number and n an integer so large that $3/4^n+3/4^{n+1}+\cdots<\epsilon/4$. If p and q are points of the same component of $M-M_n$, then $D(p, q)<\epsilon/2$ by (f). If p is a point of M_n, let R be the set of all points r of M_n such that $D_n(p, r)<\epsilon/2$. The sum V of R and all components of $M-M_n$ that have a point of R on their boundaries is an open subset of M containing p and if q is a point of V, then $D(p, q)<\epsilon$. Hence, the set of points q such that $D(p, q)>\epsilon$ is not a limit point of p.

If V is an open subset of M containing p, we shall show that there is a positive number ϵ such that $D(p, M-V)\geqq\epsilon$. Let R be an open subset of V containing p and n an integer such that $F(R, M-V)>3/4^n$. There is a positive number ϵ such that $D_n(r, s)>\epsilon$ if $F(r, s)>1/4^n$. Since each component of $M-M_n$ is of diameter less than $1/4^n$ under F, each arc in M from R to $M-V$ contains points r and s of M_n such that $F(r, s)>1/4^n$. Hence, if k is an integer bigger than n, $D_k(R\cdot M_k, [M-V]\cdot M_k)>\epsilon$. Hence, $D(p, M-V)\geqq\epsilon$.

We have shown that D satisfies conditions (1), (2), and (4). As each D_i satisfies conditions (3) and (5) and D is the limit of $D_1, D_2, \cdots$, then D satisfies these conditions. Hence, it is a convex metric for M.

THEOREM 5. *If M is an n-dimensional locally connected compact continuum and ϵ is a positive number, there is a locally connected continuum W in M such that each component of $M-W$ is of diameter less than ϵ, W is $(n-1)$-dimensional if $n>1$, and W is a dendron (acyclic continuous curve) if $n=1$.*

PROOF. An application of the Heine-Borel Theorem gives that there is an $(n-1)$-dimensional closed subset H of M such that each component of $M-H$ is of diameter less than ϵ. If $n-1>0$, H is contained in an $(n-1)$-dimensional locally connected subcontinuum of M [6, Theorem 1]. If $n-1=0$, a dendron in M contains H [7, Theorem 1].

THEOREM 6. *Each n-dimensional compact locally connected continuum has a convex metric.*

PROOF. If $n=1$, Theorem 6 follows from Theorems 4 and 5 and the fact that a dendron has a convex metric. If $n>1$, Theorem 6 follows from Theorems 4 and 5 and induction on n.

DEFINITION. A set S is said to be finite-dimensional if for each point p of S and each open subset R of S containing p there is an integer n and an open subset R' of R containing p such that the boundary of R' with respect to S is n-dimensional.

The following result may be established by using Theorems 4, 5, and 6.

THEOREM 7. *Each finite-dimensional compact locally connected continuum has a convex metric.*

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UNIVERSITY OF WISCONSIN