APPROXIMATION IN LIP (α, p)

PAUL CIVIN

Let L_p, $1<p<\infty$, denote the class of measurable functions of period 2π for which $(\int_{-\pi}^{\pi} |f(x)|^p dx)^{1/p} = M_p(f) < \infty$, and let Lip (α, p), $0<\alpha<\infty$, represent that subclass of L_p for which $(\int_{-\pi}^{\pi} |f(x+h) - f(x)|^p dx)^{1/p} = O(h^{-\alpha})$ as $h \to 0$. The object of the present note is to demonstrate the following theorem.

THEOREM. *If $f(x) \in$ Lip (α, p) and $\{P_n(x)\}$ is a sequence of trigonometric polynomials of order n such that*

$$(1) \qquad M_p(f - P_n) \leqq Kn^{-\alpha},$$

then

$$(2) \qquad \left(\int_{-\pi}^{\pi} |P_n'(x)|^p dx\right)^{1/p} \leqq \begin{cases} A(1-\alpha)^{-1}n^{1-\alpha}, & 0<\alpha<1, \\ A \log n, & \alpha = 1, \\ A(\alpha-1)^{-1}, & 1<\alpha<\infty \end{cases}$$

where in each case A depends only on α and the sequence $P_n(x)$ but not on n.

The method is that of M. Zamansky[1] [2] who obtained the corresponding results for functions in Lip α, $0<\alpha\leqq 1$.

An application of the inequality of Zygmund [3] concerning the pth mean of the derivative of a trigonometric polynomial together with the Minkowski inequality shows that if (1) and (2) are satisfied by a sequence $\{P_{n_j}\}$ with $(n_{j+1}/n_j) = O(1)$ and if $\{\lambda_n\}$ is any sequence of trigonometric polynomials of order n such that $M_p(\lambda_n) = O(n^{-\alpha})$, then the sequence $\{P_{n_j}+\lambda_n\}$ $(n = n_j, n_j+1, \cdots, n_{j+1}-1; j = 1, 2, \cdots)$ also satisfies (1) and (2). A further application of the same inequalities shows that if $\{P_n\}$ satisfies (1) and (2) and if $\{Q_n\}$ satisfies (1), then $\{Q_n\}$ also satisfies (2). The proof of the theorem is thus reduced to the exhibition of a sequence $\{P_{n_j}\}$ of trigonometric polynomials of order n_j with $(n_{j+1}/n_j) = O(1)$ such that (1) and (2) hold for $\{P_{n_j}\}$.

Let r be the smallest integer greater than $(1+\alpha)/2$ and $q = p/(p-1)$. If $f(x) \in L_p$ and

$$u(r) = \int_{-\infty}^{\infty} (\sin t/t)^{2r} dt$$

Presented to the Society, November 27, 1948; received by the editors June 21, 1948.

[1] Numbers in brackets refer to the references at the end of the note.

then

$$F_j(x) = (u(r))^{-1}\int_{-\infty}^{\infty} f(x + 2^{1-j}t)(\sin t/t)^{2r}dt$$

is a trigonometric polynomial[2] [1] of order less than $r2^j$. Let $P_{n_j}(x) = F_j(x) + n_j^{-\alpha} \cos n_j x$ for $n_j = r2^j$, so that $P_{n_j}(x)$ is a trigonometric polynomial of order[3] n_j and $(n_{j+1}/n_j) = O(1)$.

In view of the definition of r, it is possible to select β so that $\beta q > 1$ and $p(2r - \beta - \alpha) > 1$. Therefore

$$\begin{aligned}
M_p(f - P_{n_j}) &\leqq A_1\left(\int_{-\pi}^{\pi}\left(\int_{-\infty}^{\infty} | f(x) - f(x + 2^{1-j}t) | \, | \sin t/t |^{2r} dt\right)^p dx\right)^{1/p} \\
&\quad + Bn_j^{-\alpha} \\
&\leqq A_1\left(\int_{-\pi}^{\pi}\left(\int_{-\infty}^{\infty} | \sin t/t |^{\beta q} dt\right)^{p/q}\right. \\
&\quad \left. \cdot \int_{-\infty}^{\infty} | f(x) - f(x + 2^{1-j}t) |^p | \sin t/t |^{(2r-\beta)p} dt dx\right)^{1/p} \\
&\quad + Bn_j^{-\alpha} \\
&\leqq A_2\left(\int_{-\infty}^{\infty} | \sin t/t |^{(2r-\beta)p} \int_{-\pi}^{\pi} | f(x) - f(x + 2^{1-j}t) |^p dx dt\right)^{1/p} \\
&\quad + Bn_j^{-\alpha} \\
&\leqq A_3\left(\int_{-\infty}^{\infty} | \sin t/t |^{(2r-\beta)p} | 2^{1-j}t |^{\alpha p} dt\right)^{1/p} + Bn_j^{-\alpha} \\
&\leqq A_4 n_j^{-\alpha},
\end{aligned}$$

with the various A_k and B independent of the n_j.

By a technique identical with that in the preceding paragraph, it is easily seen that

$$M_p(P_{n_j} - P_{n_{j-1}}) \leqq An_j^{-\alpha}.$$

Another application of the Zygmund inequality shows that

---

[2] The proof given in [1] is for continuous $f(x)$. However, the proof is identical for $f(x) \in L_p$.

[3] A term of order n_j is also required for the completeness of the Zamansky proof [2] for $f(x) \in \text{Lip}\,\alpha$. As that proof was given, the difference in the actual and apparent order of the approximating polynomials causes the proof to fail in the case of functions $f(x)$ with "large" gaps in their Fourier series.

$$M_p(P'_{n_j} - P'_{n_{j-1}}) \leqq A n_j^{1-\alpha}.$$

The application of the Minkowski inequality to the sum

$$P'_{n_k}(x) = \sum_{j=1}^{k-1} (P'_{n_{j+1}}(x) - P'_{n_j}(x)) + P'_{n_1}(x),$$

followed by summation over j, completes the proof of the theorem.

REFERENCES

1. C. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, 1919.

2. M. Zamansky, *Sur l'approximation des fonctions continues*, C. R. Acad. Sci. Paris vol. 224 (1947) pp. 704–706.

3. A. Zygmund, *A remark on conjugate series*, Proc. London Math. Soc. vol. 3 (1932) pp. 392–400.

UNIVERSITY OF OREGON