ABSOLUTE-VALUED ALGEBRAIC ALGEBRAS

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1. **Introduction.** An *algebra* $\mathfrak{A}$ over a field $\mathfrak{F}$ is a vector space over $\mathfrak{F}$ which is closed with respect to a product xy which is linear in both x and y. The product is not necessarily associative. Every element x of $\mathfrak{A}$ generates a subalgebra $\mathfrak{F}[x]$ of $\mathfrak{A}$ and we call $\mathfrak{A}$ an algebraic algebra if every $\mathfrak{F}[x]$ is a finite-dimensional vector space over $\mathfrak{F}$.

We have shown elsewhere[1] that every absolute-valued real finite-dimensional algebra has dimension 1, 2, 4, or 8 and is either the field $\mathfrak{R}$ of all real numbers, the complex field $\mathfrak{C}$, the real quaternion algebra $\mathfrak{Q}$, the real Cayley algebra $\mathfrak{D}$, or certain isotopes without unity quantities of $\mathfrak{Q}$ and $\mathfrak{D}$. In the present paper we shall extend these results to algebraic algebras over $\mathfrak{R}$ showing that every algebraic algebra over $\mathfrak{R}$ with a unity quantity is finite-dimensional and so is one of the algebras listed above. The results are extended immediately to absolute-valued algebraic division algebras, that is, to algebras without unity quantities whose nonzero quantities form a quasigroup.

2. **Quadratic algebras.** Let $\mathfrak{F}$ be a field whose characteristic is not two and $\mathfrak{A}$ be an algebra over $\mathfrak{F}$ with a unity quantity 1. The scalar multiples $\alpha 1$ defined for α in $\mathfrak{F}$ form a subalgebra of $\mathfrak{A}$ isomorphic to $\mathfrak{F}$ and we may assume that $\mathfrak{F}$ is actually a subalgebra of $\mathfrak{A}$ whose unity element coincides with that of $\mathfrak{A}$. Then $\mathfrak{F}$ is a subalgebra of the center[2] of $\mathfrak{A}$. We shall call the elements of $\mathfrak{A}$ which are in $\mathfrak{F}$ the *scalars* of $\mathfrak{A}$ and all other elements of $\mathfrak{A}$ *nonscalars.*

In an algebra of degree two over $\mathfrak{F}$ every x is a root of an equation

$$(1)\qquad f(\xi, x) \equiv \xi^2 - 2\xi\tau(x) - \nu(x) = 0,$$

where $\tau(x)$ and $\nu(x)$ are functions on $\mathfrak{A}$ to $\mathfrak{F}$. Then every nonscalar x of $\mathfrak{A}$ determines a commutative associative algebra $\mathfrak{F}[x] = \mathfrak{F} + x\mathfrak{F}$ of order two over $\mathfrak{F}$. It is seen trivially that $f(\xi, x)$ is unique for every

Presented to the Society, September 10, 1948; received by the editors June 21, 1948.

[1] See the author's *Absolute valued real algebras*, Ann. of Math. vol. 48 (1947) pp. 495–501.

[2] The center of a finite-dimensional algebra was defined in the author's *Non-associative algebras* I, Ann. of Math. vol. 43 (1942) on page 707. The same definition was given later for rings by T. Nakayama, *Über einfache distributive Systeme unendlicher Range,* Proc. Imp. Acad. Tokyo vol. 20 (1944) p. 62 and by N. Jacobson, *Structure theory of rings without finiteness assumptions*, Trans. Amer. Math. Soc. vol. 57 (1945) p. 239.

nonscalar x of $\mathfrak{A}$ and that $\mathfrak{F}[x]=\mathfrak{F}[y]=\mathfrak{F}+y\mathfrak{F}$ where $y=x-\tau(x)$, $\tau(y)=0$,

$$y^2 = \nu(y) = \nu(x) + [\tau(x)]^2. \tag{2}$$

It will be convenient to call an element y a *square root* if y is a nonscalar and $\tau(y)=0$ so that $y^2=\nu(y)$ is in $\mathfrak{F}$.

An algebra $\mathfrak{A}$ will be called a *quadratic* algebra if $\mathfrak{A}$ has a unity quantity and $\mathfrak{F}[x]$ is a quadratic field over $\mathfrak{F}$ for every nonscalar x of $\mathfrak{A}$. Then $\mathfrak{A}$ has degree two over $\mathfrak{F}$ and every $f(\xi, x)$ is irreducible for x a nonscalar, $\nu(x)\neq 0$. Conversely an algebra $\mathfrak{A}$ with a unity quantity and degree two over $\mathfrak{F}$ is a quadratic algebra provided that $\nu(x)\neq 0$ for every nonscalar x. For if $f(\xi, x)$ were reducible we would have $f(\xi, x)=(\xi-\alpha)(\xi-\beta)$, $y(y-\beta+\alpha)=0$ for $y=x-\alpha$, y is a nonscalar such that $\nu(y)=0$. Assume throughout the remainder of this section that $\mathfrak{A}$ is a quadratic algebra.

The following lemma is due to L. E. Dickson.[3]

LEMMA 1. *Let u and v be square roots in $\mathfrak{A}$ such that* 1, *u, v are linearly independent in $\mathfrak{F}$. Then $uv+vu$ is a scalar.*

The proof is made by the computations $u^2=\alpha$, $v^2=\beta$, $(u+v)^2=\lambda(u+v)+\mu$, $(u+2v)^2=\rho(u+2v)+\sigma$ for $\alpha, \beta, \lambda, \mu, \rho, \sigma$ in $\mathfrak{F}$. Then $t=uv+vu=\lambda(u+v)+\mu-(\alpha+\beta)$, $2t=\rho(u+2v)+\sigma-\alpha-4\beta$ so that $2\lambda=\rho=2\rho$, $\lambda=0$, $uv+vu=\mu-(\alpha+\beta)$ is in $\mathfrak{F}$.

Two nonzero elements u and v of $\mathfrak{A}$ are said to be *J-orthogonal* if their Jordan product $uv+vu=0$. We call $u_1, \cdots, u_n$ *pairwise J-orthogonal* when u_i and u_j are J-orthogonal for every $i\neq j$. Then we have

LEMMA 2. *Let $u_1, \cdots, u_n$ be pairwise J-orthogonal nonscalar elements of a quadratic algebra $\mathfrak{A}$. Then* 1, *$u_1, \cdots, u_n$ are linearly independent in $\mathfrak{F}$.*

For if $t=\lambda_1u_1+\cdots+\lambda_nu_n+\lambda_0=0$ for $\lambda_0, \lambda_1, \cdots, \lambda_n$ in $\mathfrak{F}$ then $tu_i+u_it=2(\lambda_0u_i+\lambda_iu_i^2)=0$. If $\lambda_i\neq 0$ we would have $\nu(u_i)=0$ contrary to our hypothesis on $\mathfrak{A}$. Hence $\lambda_i=0$, $\lambda_0=0$ as desired.

LEMMA 3. *Let $u_1, \cdots, u_n$ be pairwise J-orthogonal square roots of $\mathfrak{A}$ and w be an element of $\mathfrak{A}$ not in $\mathfrak{B}=\mathfrak{F}+u_1\mathfrak{F}+\cdots+u_n\mathfrak{F}$. Then there exists an element b of $\mathfrak{B}$ such that $u=w-b$ is a square root of $\mathfrak{A}$ and $u_1, \cdots, u_n, u$ are pairwise J-orthogonal.*

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[3] Cf. *Linear algebras with associativity not assumed*, Duke Math. J. vol. 1 (1935) pp. 113–125.

We have $w^2=2\beta w+\gamma$ for β and γ in $\mathfrak{F}$ and we put $v=w-\beta$ and see that $v^2=\gamma+\beta^2$, v is a square root. Since v is not in $\mathfrak{B}$ the quantities 1, u_i, v are linearly independent and we apply Lemma 1 to obtain $vu_i+u_iv=\delta_i$ in $\mathfrak{F}$. Define $b_0=2^{-1}\sum_{j=1}^{n}\alpha_j^{-1}\delta_ju_j$ where $u_j^2=\alpha_j$. Then $b_0u_i+u_ib_0=\alpha_i^{-1}\delta_iu_i^2=\delta_i$. But then $(v-b_0)u_i+u_i(v-b_0)=0$ and $u_1, \cdots, u_n$, u are pairwise J-orthogonal if $u=v-b_0=w-b$ where $b=\beta+b_0$ is in $\mathfrak{B}$. Also $u^2=v^2+2^{-1}\sum_{j=1}^{n}\alpha_j^{-1}\delta_j(vu_j+u_jv)+4^{-1}\sum_{j=1}^{n}\alpha_j^{-2}\delta_j^2$ is in $\mathfrak{F}$ and our proof is complete.

We shall apply the simple results just obtained to prove the finite dimensionality of alternative *quadratic* algebras, that is, of quadratic algebras such that $x(xy)=(xx)y$, $(yx)x=y(xx)$ for every x and y of $\mathfrak{A}$. It is necessary to consider only the case where $\mathfrak{A}$ contains a subspace $\mathfrak{H}$ of dimension at least eight. Thus we may assume that $\mathfrak{A}$ contains a pair of J-orthogonal square roots u and v such that $u^2=\alpha\neq 0$ in $\mathfrak{F}$, $v^2=\beta\neq 0$ in $\mathfrak{F}$, $uv=-vu$. By the alternative law we have

$$u(uv) = \alpha v,\ (uv)u = -(vu)u = -v(uu) = -\alpha v \neq 0.$$

Then uv and u are J-orthogonal. Similarly uv and v are J-orthogonal. But then $uv\neq 0$ is a nonscalar and, by Lemma 2, $\mathfrak{A}$ contains the four-dimensional subspace $\mathfrak{Q}=\mathfrak{F}+u\mathfrak{F}+v\mathfrak{F}+uv\mathfrak{F}$ of $\mathfrak{A}$. Also $(uv)^2=\gamma+\delta uv$ for γ and δ in $\mathfrak{F}$, $u[(uv)^2]=[u(uv)](uv)=\alpha v(uv)=-\alpha v(vu)=-\alpha\beta u=\gamma u+\alpha\delta v$. It follows that $\alpha\delta=0$ and so $(uv)^2=\gamma=-\alpha\beta$. Evidently $\mathfrak{Q}$ is a linear algebra which is a subring of $\mathfrak{A}$ and it is easily verified, since all products of basal elements of $\mathfrak{Q}$ are now known, that $\mathfrak{Q}$ is an associative generalized quaternion algebra. We state the result as

LEMMA 4. *Let u and v be J-orthogonal square roots of an alternative quadratic algebra $\mathfrak{A}$. Then $\mathfrak{Q}=\mathfrak{F}+u\mathfrak{F}+v\mathfrak{F}+uv\mathfrak{F}$ is a four-dimensional subring of $\mathfrak{A}$ and is a generalized quaternion associative algebra with $u^2=\alpha$, $v^2=\beta$, $(uv)^2=-\alpha\beta$, $vu=-uv$ for $\alpha\neq 0$, $\beta\neq 0$ in $\mathfrak{F}$.*

Assume next that $\mathfrak{A}$ contains $\mathfrak{Q}$ as well as an element not in $\mathfrak{Q}$. Then Lemma 3 implies that there exists a square root which is J-orthogonal to u, v and uv. For every such square root we prove

LEMMA 5. *Let $u_1=u$, $u_2=v$, $u_3=uv$, $u_4=w$ be a set of pairwise J-orthogonal square roots of an alternative quadratic algebra $\mathfrak{A}$. Then $u_1, \cdots, u_4$, $u_5=uw$, $u_6=vw$, $u_7=(uv)w$ are pairwise J-orthogonal, $u_i^2=\alpha_i$ in $\mathfrak{F}$, $u_iu_j=\alpha_{ij}u_k$ for every $i\neq j$ where α_{ij} is in $\mathfrak{F}$ and k depends on i and j. Also $\mathfrak{B}=\mathfrak{F}+u_1\mathfrak{F}+\cdots+u_7\mathfrak{F}$ is an eight-dimensional subalgebra of $\mathfrak{A}$.*

For let $\alpha=\alpha_1$, $\beta=\alpha_2$, $\gamma=\alpha_4$. Then $(u+w)^2=\alpha+\gamma$ and by the alternative law $(\alpha+\gamma)v=(u+w)^2v=(u+w)(uv+wv)=u(uv)+w(uv)$

$+u(wv)+w(wv)=\alpha v+\gamma v-u(vw)-(uv)w$. Thus

$$(3) \qquad u(vw) = -(uv)w.$$

Similarly $(\alpha+\gamma)v=v(u+w)^2=(vu+vw)(u+w)=(vu)u+(vw)w+(vu)w+(vw)u$, $(vw)u=-(vu)w$,

$$(4) \qquad (vw)u = (uv)w = -u(vw).$$

We have proved that u and vw are J-orthogonal. By symmetry v is J-orthogonal to uw and

$$(5) \qquad v(uw) = -(uw)v = (uv)w.$$

However we may replace u, v by uv, v to see that vw and uv are J-orthogonal. Also $\mathfrak{F}+v\mathfrak{F}+w\mathfrak{F}+vw\mathfrak{F}$ is a quaternion algebra and $(vw)^2=-\beta\gamma$, vw and v are J-orthogonal.

Formula (4) may be applied with u and v replaced by uv and u to yield

$$(6) \qquad (uw)(uv) = [(uv)u]w = -\alpha vw.$$

Also $(uw)v = (vu)w = -(uv)w$. Then $(u+uw)^2(uv) = (\alpha-\alpha\gamma)uv = (u+uw)[(u+uw)(uv)]=(u+uw)(\alpha v-\alpha vw)=\alpha uv+\alpha(uw)v-\alpha u(vw)-\alpha(uw)(vw)$. But $\alpha(uw)v-\alpha u(vw)=-\alpha(uv)w+\alpha(uv)w=0$. Hence

$$(7) \qquad (uw)(vw) = \gamma uv.$$

Interchange u and v to obtain $(vw)(uw)=\gamma vu=-(uw)(vw)$. It follows immediately that the symmetry among the elements u, v, uv implies that the elements $u_1, \cdots, u_7$ are pairwise J-orthogonal. They are square roots since a product such as uw is in a quaternion algebra $\mathfrak{F}+u\mathfrak{F}+w\mathfrak{F}+uw\mathfrak{F}$ and is a square root by Lemma 4. But (4) and (7) yield $u_iu_j=\alpha_{ij}u_k$.

We finally assume that $\mathfrak{A}$ contains $\mathfrak{B}=\mathfrak{F}+u_1\mathfrak{F}+\cdots+u_7\mathfrak{F}$ and an element not in $\mathfrak{B}$. Then $\mathfrak{A}$ contains an element z which is J-orthogonal to $u_1, \cdots, u_7$. We form the product $t=(uv)(wz)$. Now $z^2=\delta\neq 0$ in $\mathfrak{F}$, $(uv)t=-\alpha\beta wz$, $[(uv)t]^2=\alpha\beta\gamma\delta\neq 0$, $t\neq 0$. However we use the quaternion algebra $\mathfrak{F}+uv\mathfrak{F}+w\mathfrak{F}+(uv)w\mathfrak{F}$ with z and apply (4) to obtain $t=-[(uv)w]z$. Then $t=[u(vw)]z$. Since z is orthogonal to u, vw and $u(vw)$ we now have $t=-u[(vw)z]=u[v(wz)]$. Apply Lemma 5 with $u_1=u$, $u_2=w$, $u_4=z$ to get wz orthogonal to u. The same values, except for $u_1=v$, yields wz orthogonal to v and similarly $u_1=uv$ implies that wz is orthogonal to uv. But then (4) yields $t=-(uv)(wz)=-t$. We have proved[4]

[4] Alternative quadratic algebras of finite dimension were determined by M. Zorn, Abh. Math. Sem. Hamburgischen Univ. vol. 9 (1933) pp. 395–402.

THEOREM 1. *Every alternative quadratic algebra is an algebra of order* $n=1, 2, 4, 8$ *over* $\mathfrak{F}$.

3. Absolute valued algebraic algebras. Let $\mathfrak{R}$ be the field of all real numbers and $\mathfrak{A}$ be an algebraic algebra over $\mathfrak{R}$. We assume the existence of a function $|a|$ on $\mathfrak{A}$ to $\mathfrak{R}$ such that $|0|=0$, $|a|>0$ if $a\neq 0$, $|ab|=|a||b|$, $|a+b|\leqq|a|+|b|$ for every a and b of $\mathfrak{A}$. Assume also that if a is in $\mathfrak{A}$ and α is any real number then $|\alpha a|=|\alpha||a|$ where $|\alpha|$ is the ordinary absolute value of α. Then we call $\mathfrak{A}$ an absolute-valued algebra.

We shall assume that $\mathfrak{A}$ has a unity quantity 1 so that $\mathfrak{R}$ may be taken to be a subalgebra of $\mathfrak{A}$ and the hypothesis $|\alpha a|=|\alpha||a|$ now becomes simply the hypothesis that $|a|$ is the ordinary absolute value whenever a is a real number. The hypothesis that $\mathfrak{A}$ is an algebra over $\mathfrak{R}$ implies that $\mathfrak{R}[x]$ is an absolute-valued real algebra generated by x and containing 1. However all real, absolute-valued algebras with a unity quantity are known,[1] they all have degree one or two and hence $\mathfrak{R}[x]$ is isomorphic to either $\mathfrak{R}$ or the field $\mathfrak{C}$ of all complex numbers. It follows immediately that $\mathfrak{A}$ is a quadratic algebra over $\mathfrak{R}$.

Every element x of $\mathfrak{A}$ now has the form $x=\alpha+\beta u$ where α and β are real and $u^2=-1$. Then $\mathfrak{R}(u)$ is isomorphic to $\mathfrak{C}$, $(x-\alpha)^2+\beta^2=0$, the roots of $f(\xi, x)=\xi^2-2\alpha\xi+\alpha^2+\beta^2$ are ρ, $\bar{\rho}$ where $\rho=\alpha+\beta i$, $\bar{\rho}=\alpha-\beta i$, $\rho\bar{\rho}=\alpha^2+\beta^2$, $|\rho|=(\alpha^2+\beta^2)^{1/2}$. It is known[3] that the minimum function of x and R_x coincide with $f(\xi, x)$ and that $|x|=|\rho| =(\alpha^2+\beta^2)^{1/2}$.

Assume that $u_1, \cdots, u_n$ are pairwise J-orthogonal quantities such that $u_i^2=-1$. Then if $x=\alpha_0+\alpha_1u_1+\cdots+\alpha_nu_n$ we have $x^2-2\alpha_0x+\alpha_0^2+\cdots+\alpha_n^2=0$. It follows that $|x|^2=\alpha_0^2+\alpha_1^2+\cdots+\alpha_n^2$.

We are now able to compute the absolute values of quantities and shall assume that $\mathfrak{A}$ contains a pair of J-orthogonal quantities u and v such that $u^2=v^2=-1$. By Lemma 3 we may write $uv=\lambda_1+\lambda_2u+\lambda_3v+\lambda_4w$ where $\lambda_1, \cdots, \lambda_4$ are real, u, v, w are pairwise J-orthogonal and $w^2=-1$. Of course when uv is in $\mathfrak{F}+u\mathfrak{F}+v\mathfrak{F}$ this will be given by $\lambda_4=0$. Then $|uv|^2=|u|^2|v|^2=1=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2$. But $|1+v|^2=2$, $|u(1+v)|^2=2=|u+uv|^2=|\lambda_1+(\lambda_2+1)u+\lambda_3v+\lambda_4w|^2=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+1+2\lambda_2$. Then $\lambda_2=0$. A similar computation of $|(1+u)v|^2$ yields $\lambda_3=0$, $\lambda_1^2+\lambda_4^2=1$. Finally $|(1+u)(1+v)|^2=|(\lambda_1+1)+u+v+\lambda_4w|^2=4=\lambda_1^2+\lambda_4^2+3+2\lambda_1$, $\lambda_1=0$, $\lambda_4=\pm 1$, $uv=\pm w$ is J-orthogonal to u and v and $(uv)^2=w^2=-1$.

Apply the argument above to the J-orthogonal quantities u and uv to see that $[u(uv)]^2=-1$ and that $u(uv)$ is J-orthogonal to both u and uv. Then $u(uv)=\xi v+\eta z$ where ξ and η are real, z is J-orthogonal to

u, v and uv, $z^2=-1$ and of course the nonexistence of such a z implies that $\eta=0$. Since $|u(wv)|=|u|^2|v|=1$ we have $\xi^2+\eta^2=1$. We form the product $(-1+u)(v+uv)=-v+uv-uv+u(uv)=-v+u(uv)$ $=(\xi-1)v+\eta z$ and see that $4=|(\xi-1)v+\eta z|^2=\xi^2+\eta^2+1-2\xi$, -2ξ $=2$, $\xi=-1$, $\xi^2+\eta^2=1+\eta^2=1$, $\eta=0$. Hence $u(uv)=-v$.

The results above imply that $\mathfrak{A}$ is alternative. Indeed the relation $x(xy)=(xx)y$ is equivalent to $(\alpha+\beta u)[(\alpha+\beta u)y]=[(\alpha+\beta u)(\alpha+\beta u)]y$ for real α and β where $u^2=-1$. But $\alpha(\alpha y)+(\beta u)\alpha y+\alpha[(\beta u)y]$ $=\alpha^2y+[(\alpha\beta)u]y+[(\beta u)\alpha]y$ and so we need only prove $u(uy)=(uu)y$. We may also write $y=\lambda+\mu u+\nu v$ where λ, μ, ν are real, $uv=-vu$ and $v^2=-1$, and the linearity of our relation in y implies that we need only prove the trivial $u(u^2)=u^2u=-u$ and the relation $u(uv)=(uu)v$ $=-v$ proved above. Similarly the relation $(yx)x=y(xx)$ is implied by $(vu)u=-(uv)u=u(uv)=-v=v(uu)$. We then apply Theorem 1 to obtain

THEOREM 2. *Let $\mathfrak{A}$ be an absolute-valued real algebraic algebra with a unity quantity. Then $\mathfrak{A}$ is a finite-dimensional algebra and is indeed the field $\mathfrak{R}$ or $\mathfrak{C}$, the quaternion algebra $\mathfrak{Q}$ or the Cayley algebra $\mathfrak{D}$.*

An algebra is called a *division algebra* if its nonzero elements form a quasigroup. For such algebras we have

THEOREM 3. *Let $\mathfrak{A}$ be an absolute valued real algebraic division algebra without a unity element. Then $\mathfrak{A}$ is an algebra of order four or eight.*

For the mappings $x\rightarrow xg=xR_g$ and $x\rightarrow gx=xL_g$ are one-to-one mappings of $\mathfrak{A}$ on $\mathfrak{A}$ for every $g\neq 0$ of $\mathfrak{A}$ and so R_g and L_g have inverses. Select g so that $|g|=1$ and take $P=R_g^{-1}$, $Q=L_g^{-1}$. Let $\mathfrak{A}_0$ be the same real vector space as $\mathfrak{A}$ and define multiplication in $\mathfrak{A}_0$ by $x\cdot y=(xP)(yQ)$. Then if $e=gg$ we have $eP=eQ=g$, $e\cdot y=g(yQ)$ $=yL_g^{-1}L_g=y$, $y\cdot e=(yR_g^{-1})g=y$ and so $\mathfrak{A}_0$ has e as its unity quantity. Since $\mathfrak{A}_0$ is the same vector space as $\mathfrak{A}$ the additive and scalar properties of $|x|$ all hold in $\mathfrak{A}_0$. But $|x\cdot y|=|xP|\cdot|yQ|=|x||y|$ since $(xP)g=x$, $|x|=|xP||g|=|xP|$, $g(yQ)=y$, $|y|=|g||yQ|=|yQ|$. Thus $\mathfrak{A}_0$ is an absolute valued algebraic algebra with a unity quantity and is finite-dimensional by Theorem 2, $\mathfrak{A}$ is finite-dimensional. However the result on the dimension of $\mathfrak{A}$ of our Theorem is known[1] for algebras.

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