BOOK REVIEW

Functional analysis and semi-groups. By Einar Hille. American Mathematical Society Colloquium Publications, vol. 31. New York, American Mathematical Society, 1948. 12+528 pp. $7.50.

This is a very interesting and important treatise, embodying several books as it were, the largest among the latter being the one actually devoted to semi-groups themselves. The treatise might be described as a "cours d'analyse" for the study of vector-valued functions, Banach norms and spectral theory; but it also contains such "classical" material as Laplace integrals, Fourier series, Hermite expansions, real functions in Euclidean spaces, and so on.

1. The treatise incorporates first of all a kind of monograph, very up-to-date but not entirely complete, on ordinary Banach spaces with a goodly emphasis on "categories" which the author avers arises from the influence of Max Zorn. This is a very valuable supplement to Banach's own manual. The author then rapidly proceeds to the development of the notions of Banach valued measurable and integrable functions with which he will afterwards operate incessantly. We bow to the author in gratitude for naming his principal integral after us. But we were surprised to find that he does not describe how the transition from the ordinary Riemann integral to Lebesque integral may in itself be derived by a process of Banach-completion of a normed vector-space at first incomplete.

2. In the second place, Chapters V and XX of the treatise constitute a most handy monograph on so-called Banach algebras. A Banach algebra (and who does not know it by now?) is a Banach space with a super-imposed ring multiplication satisfying the relation

$$\|x\cdot y\| \leqq \|x\|\cdot\|y\|.$$

The Abelian prototype is the space of complex-valued functions $f(t)$, $-\infty < t < \infty$, with the norm $\int_{-\infty}^{\infty} |f(t)|\, dt$ and the multiplication

$$h(t) = \int_{-\infty}^{\infty} f(t-u)g(u)du.$$

The author gives an account of maximal ideals, including Gelfand's fundamental theorem on numerically-valued functions of maximal ideals for commutative Banach algebras. However the author does not include Gelfand's application of his theorem to the proof of Wiener's theorem on ordinary Fourier series. This is a most regrettable omission, seeing that this application is still probably the most telling link between the theory of Banach algebras and analysis of the homespun variety. The author has theorems on "structure" and "representation" (but no positive-definite functions); he greatly stresses the efforts of the contemporary school which originated technically with Sam Perlis and N. Jacobson to avoid the addition of a ring-unit whenever one is not present at the start. In the previously described prototype of a commutative Banach algebra the missing unit would be the "Dirac function" $e=dF(t)/dt$, $F(t)=0$, $-\infty<t<0$, $F(t)=1$, $0\leqq t<\infty$. In this and all other cases the unit can be added very economically by forming the elements $\lambda e+a$. It is originally needed for the definition of "inverse" elements; but the basic relation $(a+e)(b+e)=e$ reduces to $ab+a+b=0$, and in the latter version the unit has been eliminated. It is claimed that the artificial addition of a unit would have distortive consequences, but this claim is not substantiated by any illustrative material.

3. Then there is a treatment of complex analytic functions from Banach space to Banach space. Assuming, for our discussion now, the values of the functions to be complex numbers, this is an extension of functions $f(z_1, \cdots, z_k)$ from a complex E_k to a Banach space B. In the latter case $f(a)$ is called analytic in D, $D\subset B$, if for a in D, b in B, and t complex, $f(a+bt)$ is analytic in t, "uniformly" in a, b. This analyticity is also described in terms of differentiability properties, mainly with a view to obtaining statements on the domain of convergence of what corresponds to the Taylor-expansion of $f(z_1, \cdots, z_k)$ around the origin in its diagonal arrangement $\sum_{n=0}^{\infty} P_n(z_1 \cdots z_k)$. In this entire context some very astute "category arguments" are used and the results are very interesting; but in connection with the theorem stated on page 90 it should be emphasized that it is a generalization of a classical theorem of Hartogs for functions in k complex variables, and that Hartogs' own version is presupposed known for the generalization and apparently has not been proved anew by the category argument.

4. If e is the unit and a any other element of a B-algebra, for example, an algebra of endomorphisms in some Banach space, the spectral analysis of a consists of a discussion of the expression

$$(\lambda e - a)^{-1} \tag{1}$$

considered as a function from the complex numbers λ to elements in the algebra; the resolvent set is the open λ-set for which the inverse exists, and the spectrum is its complementary set. The author indicates in what manner N. Dunford, E. R. Lorch, C. E. Rickart, A. E. Taylor and others have successfully extended and expanded previous results about operators in Hilbert space to general B-algebras. In this connection it seems that it would be worth while to investigate the nature of the expression

$$(\lambda_1 e_1 + \cdots + \lambda_k e_k - a)^{-1}$$

as a function of k complex variables $\lambda_1, \cdots, \lambda_k$, for a set of "unit-like" elements $e_1, \cdots, e_k$. For instance, if $e = e_1 + \cdots + e_k$ with $e_i e_j = \delta_{ij} e_i$, then most likely the resolvent set is a topological product of open sets in the λ_j-planes separately, and is thus of rather special structure. But even in the most general set-up, the resolvent set could not be any given open set in λ-space (due to structural requirements on domains of analyticity for $k \geqq 2$), but our best guess is that an open set is a resolvent set provided it is the set of uniform convergence of a series

$$\sum_{n=0}^{\infty} (\alpha_{n1}\lambda_1 + \cdots + \alpha_{nk}\lambda_k + \alpha_{n0})^{-1}$$

with suitable numerical coefficients $\alpha_{\nu j}$.

For fixed λ but variable a the expression (1) is an analytic function from a Banach algebra to itself. For such functions in general, E. R. Lorch has given a definition of analyticity and of the Cauchy integral $\int_C f(x)dx$, for x and $y = f(x)$ both in a B-algebra. From the author's standpoint the most important instances of such functions are either the resolvent (1), or an exponential e^{-tx} and its inverse, $\log x \equiv \int (1/x)dx$, both of which are fundamental to the very definition of a semi-group. They are special cases of general numerical functions of x, x in B, the latter being then roughly speaking a power series $\sum \alpha_n x^n$ with *numerical* coefficients α_n. For a non-commutative B-algebra, more general functions $f(x)$ have hardly been approached, even in the finite-dimensional case, and research in this direction would be most desirable.

5. Two of the most appealing sections of the treatise are Chapters VI and VII, insofar as they are devoted to an analysis of subadditive functions and to a purely geometric description of the boundary of Euclidean semimodules (that is, additive Abelian semi-groups), in particular to those having the origin of the space on its boundary (angular semi-modules). The author's digest of the existing material is most useful. We are taking the liberty of replenishing his list of references by quoting two items of our own in which angular semi-modules have been explicitly introduced for the purposes of Fourier analysis and Cauchy formulas, namely: *Group invariance of Cauchy's formula in several variables*, Ann. of Math. vol. 45 (1944) p. 686, and *Bound-*

ary values of analytic functions in several variables, Ann. of Math. vol. 45 (1944) p. 708.

6. A semi-group is a solution $f(\xi)$ of the functional equation

$$f(\xi_1 + \xi_2) = f(\xi_1)f(\xi_2) \tag{2}$$

which is defined from the half-line $0 < \xi < \infty$, or a larger semi-module, to a Banach algebra, preferably an algebra of bounded linear transformations of a Banach space into itself; in the latter case it is better to write the above in the form:

$$T(\xi_1 + \xi_2)x = T(\xi_1)[T(\xi_2)x]. \tag{3}$$

With regard to the numerical solutions of (2) one may make two statements: (i) if a solution has any kind of "approximate" continuity, for example if it is Lebesgue measurable, then it automatically is "strictly" continuous; and (ii) if it is so continuous, it has the trivial form

$$f(\xi) = e^{a\xi}. \tag{4}$$

Now, in the non-numerical case, statement (i) still remains in force on the whole, and we shall not discuss it any further. But statement (ii) is subject to serious qualifications. First of all, (4) must be immediately amended to read

$$f(\xi) = je^{a\xi} \tag{5}$$

where $j = j^2$ is any idempotent element. Secondly, the existence of an element a which satisfies (5), or of an *un*bounded transformation A which satisfies

$$T(\xi)x = j[e^{A\xi}x] \tag{6}$$

is predicated on a foreknowledge that the limits

$$\lim_{\xi \to +0} f(\xi) = f(+0), \qquad \lim_{\eta \to 0} \frac{f(\eta) - f(+0)}{\eta}$$

do exist in some appropriate manner, and the manner in which the latter exist greatly conditions the way in which the equations (5) and (6) are to be interpreted.

Also, due to the possible generality of the spectrum of a (or A), the actual complex-analytic behaviour of $f(\xi)$ is not that of a simple exponential monomial as the simplicity of the symbol (5) might deceptively suggest. Rather it is that of an analytic function defined only in an angular sector, and exhibiting there the behaviour of general functions of exponential type, complete with indicator diagram à la Pincherle-Pólya, Laplace-transforms, and such like. The most notable features by which to distinguish it from a function of exponential type in general are as follows. If $f(\xi)$ can be appropriately continued analytically into a complex neighborhood of some finite real interval $\alpha \leqq \xi \leqq \beta$ then it can be so continued into an entire semi-module, and for $\alpha = 0$ this is even an angular semi-module. Furthermore, depending on the set-up, the function $f(\xi)$ has usually some kind of limit behaviour at the termini of the sector, for both $\xi \rightarrow +0$ and $\xi \rightarrow +\infty$. The body of statements concerning such limit behaviour is what is commonly called "ergodic theory"; and with some adaptation of approach a certain part of the "Tauberian theory" may also be thus included. In discussing the latter "theories" the author does not draw on the concepts of "partial ordering" except for a brief description of a paper by the reviewer on p. 381. This leads to a tangible curtailment of viewpoint when discussing such topics as the spectral resolution of a Hermitian operator in Hilbert space, or G. D. Birkhoff's version of the ergodic theorem. But the author has knowingly chosen his viewpoints, as undoubtedly was his privilege.

7. Finally, under the designation of "special semi-groups" the author has assembled several topics in "concrete" analysis each involving a semi-group and falling under the following general scheme.

Take a system of functions $\{\phi_n(x)\}$ on some point set S, the index n being discrete or continuous. Next take a vector space of functions on S each admitting an expansion

$$f(x) \sim \sum_n a_n \phi_n(x)$$

or

$$f(x) \sim \int a_n \phi_n(x) dn$$

with suitable constant coefficients $\{a_n\}$. Now, take a system of nonnegative exponents $\{\lambda_n\}$ and for $\xi > 0$ introduce the transformations

$$T(\xi)f \sim \sum e^{-\xi\lambda_n} a_n \phi_n(x).$$

They obviously form a semi-group. Furthermore, if the system $\{\phi_n(x)\}$ is orthonormal with respect to a suitable measure on S, we have the kernel representation

$$T(\xi)f \sim \int_S K(\xi, x, y) f(y) d\delta y \tag{7}$$

where

$$K(\xi, x, y) \sim \sum_n e^{-\xi\lambda_n}\phi_n(x)\phi_n(y). \tag{8}$$

Now, one may envisage admitting an "arbitrary" sequence of exponents $\{\lambda_n\}$ and attempt to describe the corresponding arbitrary semigroup $T(\xi)$ and its associated kernel (8). This is an interesting enough problem, but a somewhat artificial one. A much more natural set-up arises if the exponents $\{\lambda_n\}$ and the functions $\{\phi_n(x)\}$ are in some specific relationship to each other, to wit, if they are the eigenfunctions and eigenvalues of some suitable differential equation

$$A_x\phi = \lambda\phi.$$

In such a case the transformed function (7) is the solution of the associated "heat equation"

$$A_xf(x, \xi) = -\lambda\frac{\partial f(x, \xi)}{\partial\xi}$$

with the initial value $f(x, 0) = f(x)$. Or, if we have

$$A_\lambda\phi = \lambda^2\phi,$$

then the function is a solution of the "Laplace equation"

$$A_xf(x, \xi) = \frac{\partial^2 f(x, \xi)}{\partial\xi^2}$$

again with the same initial value, and under the condition of remaining bounded for $0 < \xi < \infty$. The nature of the kernel (Gauss-kernel, Abel-kernel, and so on) is then pushed sharply into the foreground and its behaviour closely reflects the nature of the operator $A_x\phi$ under discussion. For instance, roughly speaking, whenever the operator is a positive-definite one, the kernel will be a non-negative function and only then. In broaching these topics the author has by no means exhausted the existing fund of information, whether actual or potential, or delineated all existing interconnections. But he has performed the important service of assembling and sifting some prerequisite material from an absorbing field of study of ever widening interest.

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