ON CONTINUOUS CURVES IRREDUCIBLE ABOUT COMPACT SETS

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Some years ago, Leo Zippin [4][1] raised the following question: Given a compact set K in a locally connected complete metric space, S, it is known that S contains a compact continuum M irreducible about K. What conditions on K are necessary and sufficient that M may be chosen locally connected? He showed that a necessary condition is that K be a curve set; that is, the nondegenerate components of K are locally connected and form a null sequence. For K 1-dimensional, he proved the condition is also sufficient, and it was conjectured that it is always sufficient. Later Martin Ettlinger [3] gave other results tending to support this. In this note I give a partial solution.

THEOREM. *If K is a compact curve set in a convex metric space S, then S contains a locally connected compact continuum irreducible about K.*

The recent announcement by R. H. Bing [1] of a proof that every finite-dimensional Peano space has a convex metric makes this result appear less special than it might seem otherwise.

PROOF. Zippin has shown in [4] that the proposition need be proved only for the case where K is a special curve set, which is a curve set with only a countable number of components, all but one being points forming a sequence converging to that one. His argument is stated for S compact, but, however, is valid under our hypothesis. Hence, we may suppose K is the sum of a sequence $\{x_n\}$ of distinct points and a locally connected continuum C. The sequence x_n can be chosen so that $d(x_n, C) \geqq d(x_{n+1}, C)$. For each n, let y_n denote a point of C such that $d(x_n, y_n) = d(x_n, C)$. Let T_1 denote a straight-line interval from x_1 to y_1. Let x_{n_1} be the first point of x_n not in T_1, and let T_2 denote a straight-line interval joining x_{n_1} and y_{n_1}. If $T_1 \cdot T_2$ is not empty, replace the interval of T_2 irreducible about $y_{n_1} + T_1 \cdot T_2$ by the arc of T_1 irreducible about $y_1 + T_1 \cdot T_2$, and let T_2 now denote this new arc. It is easy to see from the triangle inequality and the convexity of T_1 and the original T_2 that this does not change the length of T_2. We now have that $T_1 \cdot T_2$ is either empty or is connected and contains y_1. Let x_{n_2} be the first point of x_n not in

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[1] Numbers in brackets refer to the bibliography.

T_1+T_2, and let T_3 denote a straight-line interval from x_{n_2} to y_{n_2}. If $(T_1+T_2)\cdot T_3$ is not empty, a modification of T_3 similar to that of T_2, without changing the length of T_3, permits us to assume that $(T_1+T_2)\cdot T_3$ is connected and contains one of y_1 and y_{n_1}. Continue the process indefinitely. We obtain a sequence T_n of straight-line intervals such that (1) each point x_k is contained in one; (2) each has a point of $\{x_n\}$ for one end point, and a point of $\{y_n\}$ for the other, that being its only point in C; (3) each is a shortest path between its end point in $\{x_n\}$ and C; and (4) the intersection of each two is either empty or is connected and intersects C. Let M denote $C+\sum T_n$; then M is obviously connected, and irreducibly connected about $C+\sum x_n$. We need only verify local connectivity. If x is a point of $M-C$, then only a finite number of sets T_n have points more than $2^{-1}d(x, C)$ from C, so that x is in an open subset of M which is the sum of a finite number of locally connected sets. Hence M is locally connected in $M-C$. If U is an open subset of M intersecting C, and H is a component of U, then $H\cdot C$ is open in $U\cdot C$, being the sum of components of $U\cdot C$. Also $H\cdot(M-C)$ is open in U, being the sum of components of $U\cdot(M-C)$. If a point y of $H\cdot C$ were a limit point of $(U-H)\cdot M$, it would be in the limiting set of a convergent sequence T_n' of sets T_n. But then it is the sequential limiting point of $T_n'\cdot C$, which contradicts the fact that $H\cdot C$ is open in C. Hence every component of U is open in M, which completes our proof.

Corollary. *If the compact curve set K is finite-dimensional and lies in a connected and locally arcwise connected metric space, S, then S contains a compact locally connected continuum irreducible about C.*

Proof. It is well known[2] that under our hypothesis S contains a compact locally connected continuum M containing K which is obtained by adding a countable number of arcs to K, and which is therefore also finite-dimensional. The corollary then follows from Bing's result and the theorem.

The reader interested in metric geometry will notice that the hypotheses of the theorem could be weakened or altered by assumptions about length, quasi-convexity, and so on. Such changes would have required more space without altering the essential nature of the argument. For such concepts, the reader is referred to Blumenthal [2]. In this connection, it seems to me that a large part of the possible interest in this note is due to its exploitation of metric methods in topology, which has not yet been systematically studied.

[2] See the references in [4], or the proof for Moore spaces in [3].

Added in proof. Since this note was submitted, E. E. Moise and Bing have each announced solutions of the convexification problem, so that this note settles Zippin's question.

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