ORTHOGONALITY PROPERTIES OF C-FRACTIONS

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1. **Introduction.** It has been indicated in the work of Tchebichef and Stieltjes that the denominators $D_p(z)$ of the successive approximants of a J-fraction

$$(1.1)\qquad \frac{b_0}{d_1+z} - \frac{b_1}{d_2+z} - \frac{b_2}{d_3+z} - \cdots$$

constitute a sequence of orthogonal polynomials. The orthogonality relations which exist between the $D_p(z)$ may be expressed in the following way (cf. [4, 7]).[1] Let S' be defined as the operator which replaces every z^p by c_p in any polynomial upon which it operates, where the $\{c_p\}$ are a given sequence of constants. Then the orthogonality relations

$$(1.2)\qquad S'(D_p(z)D_q(z))\begin{cases} = 0 & \text{for } p \neq q, \\ \neq 0 & \text{for } p = q, \end{cases}$$

hold relative to the operator S' and the sequence $\{c_p\}$. The polynomials $D_p(z)$ are given recurrently by the formulas $D_0(z)=1$, $D_p(z)=(d_p+z)D_{p-1}(z)-b_{p-1}D_{p-2}(z)$, $p=1, 2, \cdots$ $(D_{-1}(z)=0)$.

In this paper orthogonality relations similar to (1.2) are developed for the polynomials $B_p^*(z)$ which are derived from the denominators $B_p(z)$ of the successive approximants of a C-fraction

$$(1.3)\qquad 1 + \frac{a_1z^{\alpha_1}}{1} + \frac{a_2z^{\alpha_2}}{1} + \frac{a_3z^{\alpha_3}}{1} + \cdots,$$

where the a_p denote complex numbers and the α_p positive integers (cf. [3]). In fact, conditions (1.2) for a certain J-fraction are shown to be a specialization of the orthogonality relations for a C-fraction. Furthermore, necessary and sufficient conditions are obtained for the unique existence of the polynomials $B_p^*(z)$ in terms of the sequence $\{c_p\}$ (Theorem 2.2).

2. **Orthogonal polynomials constructed from the denominators of the approximants of a C-fraction.** Let $A_p(z)$ and $B_p(z)$ denote the numerator and denominator, respectively, of the pth approximant

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

of a C-fraction (1.3). The recurrence formulas

$$
(2.1)\quad
\begin{aligned}
A_0(z) &= 1, \qquad A_1(z) = 1 + a_1 z^{\alpha_1},\\
A_{p+1}(z) &= A_p(z) + a_{p+1} z^{\alpha_{p+1}} A_{p-1}(z),\\
B_0(z) &= 1, \qquad B_1(z) = 1,\\
B_{p+1}(z) &= B_p(z) + a_{p+1} z^{\alpha_{p+1}} B_{p-1}(z), \qquad p = 1, 2, \cdots,
\end{aligned}
$$

show that $A_p(z)$ and $B_p(z)$ are polynomials of the form

$$
(2.2)\quad
\begin{aligned}
A_p(z) &= \gamma_0^{(p)} + \gamma_1^{(p)} z + \cdots + \gamma_{s_p}^{(p)} z^{s_p},\\
B_p(z) &= 1 + \beta_1^{(p)} z + \cdots + \beta_{t_p}^{(p)} z^{t_p}.
\end{aligned}
$$

From the determinant formula

$$A_p(z)B_{p-1}(z) - A_{p-1}(z)B_p(z) = (-1)^p a_1 a_2 \cdots a_{p+1} z^{\alpha_1+\alpha_2+\cdots+\alpha_{p+1}},$$

it follows that there is determined uniquely a power series

$$(2.3)\qquad P(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

such that the power series expansion for $A_p(z)/B_p(z)$ agrees term by term with the power series $P(z)$ up to the term involving $z^{\alpha_1+\alpha_2+\cdots+\alpha_{p+1}}$, that is,

$$P(z)B_p(z) - A_p(z) = (-1)^p a_1 a_2 \cdots a_{p+1} z^{\alpha_1+\alpha_2+\cdots+\alpha_{p+1}}.$$

This uniquely determined power series is called the corresponding power series.

In [3] it was shown that the algorithm

$$
(2.4)\quad (c_n, c_{n-1}, c_{n-2}, \cdots)
\begin{pmatrix} 1 \\ \beta_1^{(p)} \\ \beta_2^{(p)} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}
= \begin{cases}
0 \text{ if } \alpha_0 + \alpha_1 + \cdots + \alpha_p < n < \alpha_0 + \alpha_1 + \cdots + \alpha_{p+1},\\
(-1)^p a_1 a_2 \cdots a_{p+1} \text{ if } n = \alpha_0 + \alpha_1 + \cdots + \alpha_{p+1},\\
p = 0, 1, \cdots \ (\alpha_0 = 0),
\end{cases}
$$

combined with formulas (2.1), gives the a_p and α_p of (1.3) corresponding to the power series $P(z)$ (2.3). Conversely, the coefficients c_p of the power series expansion $P(z)$ are determined by (2.1) and (2.4) when the corresponding C-fraction (1.3) is given.

Let the polynomials $B_p^*(z)$ be defined as follows:

$$(2.5)\qquad \begin{aligned} &B_p^*(z) = z^{n_p}B_p(1/z) = z^{n_p} + \beta_1^{(p)}z^{n_p-1} + \beta_2^{(p)}z^{n_p-2} + \cdots, \\ &\qquad\qquad \text{where } n_p = \begin{cases} s_p & \text{if} \quad s_p \geqq t_p, \\ t_p & \text{if} \quad s_p < t_p, \ p = 0, 1, \cdots. \end{cases} \end{aligned}$$

Let S be defined as the operator which replaces every z^p by c_{p+1} in any polynomial upon which it operates, and let the sequence

$$(2.6)\qquad \{c_p\}: c_1, c_2, c_3, \cdots, \qquad (c_0 = 1, c_i = 0, i < 0),$$

be the coefficients of the power series (2.3) corresponding to the C-fraction (1.3). From the algorithm (2.4)

$$(2.7)\qquad \begin{aligned} &\text{(i)} \quad S(z^kB_p^*(z)) \begin{cases} = 0 & \text{for} \quad k = 0, 1, \cdots, \sum_{i=1}^{p+1} \alpha_i - n_p - 2, \\ \neq 0 & \text{for} \quad k = \sum_{i=1}^{p+1} \alpha_i - n_p - 1; \end{cases} \\ &\text{(ii)} \quad S(z^kB_p^*(z)) = (-1)^p a_1a_2 \cdots a_{p+1} \quad \text{for} \quad k = \sum_{i=1}^{p+1} \alpha_i - n_p - 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 0, 1, \cdots. \end{aligned}$$

Let $B_q^*(z) = z^{n_q} + \beta_1^{(q)}z^{n_q-1} + \beta_2^{(q)}z^{n_q-2} + \cdots$ be one of the polynomials defined by (2.5). From (2.7) the following orthogonality relations hold between $B_p^*(z)$ and $B_q^*(z)$ relative to the operator S and the sequence $\{c_p\}$:

$$(2.8)\qquad S(B_p^*(z)B_q^*(z)) = \begin{cases} 0 \qquad \text{for} \quad n_q = 0, 1, \cdots, \sum_{i=1}^{p+1} \alpha_i - n_p - 2, \\ (-1)^p a_1a_2 \cdots a_{p+1} \neq 0 \\ \text{for} \quad n_q = \sum_{i=1}^{p+1} \alpha_i - n_p - 1, \quad p = 0, 1, \cdots. \end{cases}$$

Consequently the following theorem holds.

THEOREM 2.1. *Let the C-fraction* (1.3) *be given. By the algorithm* (2.4) *the sequence* $\{c_p\}$ (2.6) *of coefficients of the corresponding power series* (2.3) *can be computed. Then the polynomials* $B_p^*(z)$, *which are found by* (2.5) *from the denominators* $B_p(z)$ *of the successive approximants of* (1.3), *satisfy the orthogonality relations* (2.8) *relative to the operator* S *and the sequence* $\{c_p\}$.

From formulas (2.4) or conditions (2.7) there follows directly a condition necessary for the existence of the polynomials $B_p(z)$ and consequently the polynomials $B_p^*(z)$, namely, that

$$\Delta(0, 0) \neq 0,\ \Delta(0, i) = 0,\ i = 1, 2, \cdots, s_1 - 1,\ \Delta(0, s_1) \neq 0;$$
$$\Delta(t_p - 1, s_p) \neq 0,\ \Delta(t_p - 1 + i, s_p + i) = 0,$$
$$(2.9) \qquad i = 1, 2, \cdots, \sum_{i=1}^{p+1} \alpha_i - t_p - s_p - 1,$$
$$\Delta\left(\sum_{i=1}^{p+1} \alpha_i - s_p - 1, \sum_{i=1}^{p+1} \alpha_i - t_p\right) \neq 0, \qquad p = 1, 2, \cdots,$$

where

$$\Delta(j, k) = \begin{vmatrix} c_{k-j}, & c_{k-j+1}, & \cdots, & c_k \\ c_{k-j+1}, & c_{k-j+2}, & \cdots, & c_{k+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_k, & c_{k+1}, & \cdots, & c_{k+j} \end{vmatrix}, \qquad j, k = 0, 1, \cdots$$

(cf. [**3**, (3.2)]). In fact, the coefficients $a_p \neq 0$ of (1.3) may be found in terms of the determinants Δ provided conditions (2.9) hold.

Conversely, given a sequence $\{c_p\}$ for which conditions (2.9) hold, one can construct a unique system of polynomials $B_p^*(z)$ such that (2.7) hold, and consequently (2.8), and the $B_p^*(z)$ are given by formulas (2.1), (2.5), and (2.7) (ii). For, by conditions (2.9), $\Delta(0, 0) = c_0 = 1$, $\Delta(0, i) = 0$, $i = 1, \cdots, s_1 - 1$, or $c_1 = \cdots = c_{s_1-1} = 0$, and $\Delta(0, s_1) = c_{s_1} \neq 0$, it follows that $s_1 = \alpha_1$, since $c_{\alpha_1} \neq 0$. Then equations (2.7) (i) hold, that is, $S(z^k B_0^*(z)) = 0$, $k = 0, \cdots, \alpha_1 - 2$; if $B_0^*(z) = B_0(z) = 1$, and $S(z^{\alpha_1 - 1} B_0^*(z)) = a_1$ if $c_{\alpha_1} = a_1$. Thus α_1 and a_1 are determined. Suppose now $B_1^*(z) = z^{n_1} + \beta_1^{(1)} z^{n_1 - 1} + \beta_2^{(1)} z^{n_1 - 2} + \cdots = z^{\alpha_1}$ is computed by (2.1) and (2.5). By the algorithm (2.4)

$$(2.10) \qquad \sum_{i=0}^{\alpha_1} c_{\alpha_1 - i + j} \beta_i^{(1)} = 0, \qquad j = 1, \cdots, \alpha_2 - 1,$$

$$(2.11) \qquad \sum_{i=0}^{\alpha_1} c_{\alpha_1 + \alpha_2 - i} \beta_i^{(1)} = - a_1 a_2.$$

These equations are exactly conditions (2.7) for $n_1 = \alpha_1$. But since equations (2.7)(i) may be solved for the coefficients $\beta^{(1)}$ uniquely because (2.9) hold, the polynomial $B_1^*(z)$ as determined by (2.1) and (2.5) is the unique polynomial which satisfies (2.7). The value of a_2 may be determined from (2.11) and the value of α_2 from (2.9). In exactly the same way one may show for $p = 2, 3, \cdots$ that, from a given sequence $\{c_p\}$ for which conditions (2.9) hold, unique polynomials $B_p^*(z)$ may be found by (2.1), (2.5), and (2.7)(ii) such that (2.7) hold and consequently conditions (2.8). This completes the proof of the following theorem.

THEOREM 2.2. *Let $\{c_p\}$ be a given sequence of constants. There exists a unique sequence of polynomials $B_p^*(z)$ (2.5) which satisfy conditions (2.7) and consequently relations (2.8) relative to the operator S and the sequence $\{c_p\}$ if and only if conditions (2.9) hold. These polynomials are determined by the recurrence formulas (2.1), (2.5), and (2.7) (ii).*

The polynomials $B_p(z)$ (2.1) are the denominators of the successive approximants of the C-fraction (1.3) corresponding to the power series (2.3) with coefficients c_p.

If conditions (2.9) hold for $p=1, \cdots, m$, then a finite sequence of orthogonal polynomials $B_p^*(z)$ may be constructed. On the other hand, if (2.9) hold for all values of m, there exists an infinite sequence of orthogonal polynomials $B_p^*(z)$.

3. **Special *C*-fractions.** The *C*-fraction (1.3) (and its corresponding power series) is called regular if all of its approximants are Padé approximants (cf. [5]). In [3] necessary and sufficient conditions for regularity are found in terms of the s_p and t_p (cf. (2.2)). There is a special class of regular *C*-fractions called α-regular, in which the condition of regularity depends only upon the exponents α_p, that is, the α_p must satisfy the relations

$$(3.1)\qquad \begin{aligned} &\alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1} = s_{2p+1} \geqq \omega + 1 + t_{2p+1},\\ &\alpha_0 + \alpha_2 + \cdots + \alpha_{2p} \quad = t_{2p} \geqq s_{2p} - \omega, \qquad p = 0, 1, \cdots, \end{aligned}$$

where ω is an integer. In [3] it is shown that necessary and sufficient for α-regularity are the conditions

$$(3.2)\qquad \begin{aligned} &\Delta(i, \omega + 1 + i) = 0,\ i = 0, 1, \cdots, h_1 - \omega - 2,\\ &\Delta(h_1 - \omega - 1, h_1) \neq 0;\\ &\Delta(h_p - \omega - 2, h_p) \neq 0,\ \Delta(h_p - \omega - 1 + i, h_p + 1 + i) = 0\\ &\qquad\qquad\qquad i = 0, 1, \cdots, g_p - h_p + \omega - 1,\\ &\Delta(g_p - 1, g_p + \omega + 1) \neq 0;\\ &\Delta(g_p - 1, g_p + \omega) \neq 0,\ \Delta(g_p + i, g_p + \omega + 1 + i) = 0,\\ &\qquad\qquad\qquad i = 0, 1, \cdots, h_{p+1} - g_p - \omega - 2,\\ &\Delta(h_{p+1} - \omega - 1, h_{p+1}) \neq 0, \qquad p = 1, 2, \cdots, \end{aligned}$$

where $g_0=0$, $g_p=\alpha_2+\alpha_4+ \cdots +\alpha_{2p}$, $h_p=\alpha_1+\alpha_3+ \cdots +\alpha_{2p-1}$. Conditions (3.2) reduce to (2.9) when s_p and t_p satisfy (3.1).

For α-regular continued fractions, let the polynomials $B_p^*(z)$ be defined as follows:

$$(3.3)\qquad B^*_{2p+1}(z) = z^{h_{p+1}}B_{2p+1}(1/z),\quad B^*_{2p}(z) = z^{g_p+\omega}B_{2p}(1/z),\qquad p = 0, 1, \cdots .$$

Then the following theorem is a corollary to Theorem 2.2.

THEOREM 3.1. *Let* $\{c_p\}$ *be a given sequence of constants which are the coefficients of an* α*-regular power series. There exists a set of unique polynomials* $B^*_p(z)$ (3.3) *which satisfy the orthogonality relations*

$$(3.4)\qquad \begin{gathered} S(B^*_{2p+1}(z)B^*_q(z)) = \begin{cases} 0 & \text{for } n_q < g_{p+1} - 1, \\ (-1)^{2p+1}a_1a_2 \cdots a_{2p+2} \neq 0 & \text{for } n_q = g_{p+1} - 1; \end{cases} \\ S(B^*_{2p}(z)B^*_q(z)) = \begin{cases} 0 & \text{for } n_q < h_{p+1} - \omega - 1, \\ (-1)^{2p}a_1a_2 \cdots a_{2p+1} \neq 0 & \text{for } n_q = h_{p+1} - \omega - 1,\ p = 0, 1, \cdots , \end{cases} \end{gathered}$$

relative to the operator S *and the sequence* $\{c_p\}$. *These polynomials are determined by the recurrence formulas* (2.1), (3.3), *and the relations*

$$(3.5)\qquad \begin{aligned} S(z^iB^*_{2p+1}(z)) &= (-1)^{2p+1}a_1a_2 \cdots a_{2p+2} \quad \text{for } i = g_{p+1} - 1, \\ S(z^iB^*_{2p}(z)) &= (-1)^{2p}a_1a_2 \cdots a_{2p+1} \\ &\qquad \text{for } i = h_{p+1} - \omega - 1,\ p = 0, 1, \cdots . \end{aligned}$$

The polynomials $B_p(z)$ (2.1) *are the denominators of the successive approximants of an* α*-regular C-fraction.*

It may be remarked that for an α-regular C-fraction the orthogonality relations (3.4) depend entirely on the exponents α_p of the C-fraction and the integer ω, and are independent of the coefficients a_p.

A second special case of the orthogonality relations (2.8) is obtained from the denominators of the approximants of a certain continued fraction (1.1). If one specializes the C-fraction (1.3) by putting $\alpha_i = 1$, $i = 1, 2, \cdots$, and then replacing z by $1/z$, the even part (cf. [5, p. 201]) of the resulting continued fraction after certain equivalence transformations is

$$1 + \frac{a_1}{a_2 + z} - \frac{a_2a_3}{(a_3 + a_4) + z} - \frac{a_4a_5}{(a_5 + a_6) + z - \cdots}.$$

This is a J-fraction for which the orthogonality relations (2.8) reduce to the known conditions (1.2).

4. **Analogue to the Christoffel-Darboux formula.** The denominators D_p of the successive approximants of a J-fraction (1.1) are connected by the identity (cf. [1], also [2])

$$D_0(z)D_0(w) + D_1(z)D_1(w) + \cdots + D_p(z)D_p(w) = \frac{D_{p+1}(z)D_p(w) - D_p(z)D_{p+1}(w)}{z - w}, \qquad p = 0, 1, \cdots .$$

A similar relation for the denominators of the approximants of a C-fraction is shown in the following theorem.

THEOREM 4.1. *The denominators B_p of the approximants of a C-fraction* (1.3) *are connected by the identity*

$$\begin{aligned} &B_{p+1}(z)B_p(w) - B_p(z)B_{p+1}(w) \\ &\quad = a_{p+1}B_{p-1}(z)B_{p-1}(w)\left[z^{\alpha_{p+1}} - w^{\alpha_{p+1}}\right] \\ &\quad + a_{p+1}a_pB_{p-2}(z)B_{p-2}(w)\left[z^{\alpha_{p+1}}w^{\alpha_p} - w^{\alpha_{p+1}}z^{\alpha_p}\right] \\ &\quad + a_{p+1}a_pa_{p-1}B_{p-3}(z)B_{p-3}(w)\left[z^{\alpha_{p+1}}w^{\alpha_p}z^{\alpha_{p-1}} - w^{\alpha_{p+1}}z^{\alpha_p}w^{\alpha_{p-1}}\right] \\ &\quad + \cdots + a_{p+1}a_p \cdots a_2B_0(z)B_0(w)\left[z^{\alpha_{p+1}}w^{\alpha_p} \cdots \right. \\ &\qquad\qquad \left. - w^{\alpha_{p+1}}z^{\alpha_p} \cdots \right]. \end{aligned} \tag{4.1}$$

PROOF. (4.1) is derived from the recurrence formula (2.1) for B_p. For $B_{p+1}(z)B_p(w) - B_p(z)B_{p+1}(w) = B_p(w)\left[B_p(z) + a_{p+1}z^{\alpha_{p+1}}B_{p-1}(z)\right] - B_p(z)\left[B_p(w) + a_{p+1}w^{\alpha_{p+1}}B_{p-1}(w)\right] = a_{p+1}\left[z^{\alpha_{p+1}}B_p(w)B_{p-1}(z) - w^{\alpha_{p+1}}B_p(z)B_{p-1}(w)\right] = a_{p+1}B_{p-1}(z)B_{p-1}(w)\left[z^{\alpha_{p+1}} - w^{\alpha_{p+1}}\right] + a_{p+1}a_p \cdot \left[z^{\alpha_{p+1}}w^{\alpha_p}B_{p-1}(z)B_{p-2}(w) - w^{\alpha_{p+1}}z^{\alpha_p}B_{p-1}(w)B_{p-2}(z)\right]$. By a repetition of this process, formula (4.1) is ultimately obtained.

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