ON THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER INVARIANT UNDER CONTACT TRANSFORMATIONS

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In the classical Lie theory it is shown how to construct a differential equation invariant under a given group, and how to solve an equation when a group leaving the equation invariant is known. However, little is said about the problem of determining the group for a given differential equation, which is by far the most interesting problem.

In the present paper, necessary and sufficient conditions for the existence of an infinitesimal contact transformation leaving a given equation invariant are determined along with the general form of the characteristic function of the group. It will also be shown how to reduce, by a proper change of variables, the infinitesimal contact transformation to a point transformation. This enables one to solve the transformed differential equation by Lie's methods. Passing back to the original variables, a new differential equation is obtained which combined with the original equation gives its solution in parametric form.

Let

$$Bf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \pi \frac{\partial f}{\partial p}$$

be the symbol of the infinitesimal contact transformation leaving invariant the differential equation $u=F(v)$, with $u=u(x, y, p)$, $v=v(x, y, p)$, $p=dy/dx$, and F such that the equation $G(x, y, p) =u-F(v)=0$ satisfies the various conditions for the existence of solutions (but otherwise arbitrary). Throughout this paper we shall assume that:

(A) Both u and v have first derivatives with respect to x, y and p, at least in some region R of the (x, y, p)-space.

(B) The Jacobians

$$J_1 = \frac{\partial(u, v)}{\partial(y, p)}, \qquad J_2 = \frac{\partial(u, v)}{\partial(p, x)}, \qquad J_3 = \frac{\partial(u, v)}{\partial(x, y)}$$

have in R derivatives of the first and second orders, while J_1 and J_2 have also derivatives of the third order with respect to x, y and p,

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as are involved in the discussion.

(C) The functions u and v are not in involution, that is,

$$[uv] = \begin{vmatrix} u_p & u_x + pu_y \\ v_p & v_x + pv_y \end{vmatrix} = J_2 - pJ_1 \not\equiv 0.$$

Since u and v are to be invariants under Bf they will satisfy the partial differential equations

$$\xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \pi \frac{\partial u}{\partial p} = 0,$$

$$\xi \frac{\partial v}{\partial x} + \eta \frac{\partial v}{\partial y} + \pi \frac{\partial v}{\partial p} = 0,$$

from which we obtain

$$\frac{\xi}{\partial(u, v)/\partial(y, p)} = \frac{\eta}{\partial(u, v)/\partial(p, x)} = \frac{\pi}{\partial(u, v)/\partial(x, y)} = \sigma,$$

$\sigma = \sigma(x, y, p)$ being the common ratio. This can be written

$$\xi = \sigma J_1, \qquad \eta = \sigma J_2, \qquad \pi = \sigma J_3. \tag{1}$$

If W is the so-called characteristic function of the infinitesimal contact transformation, we have also

$$W = p\xi - \eta = \sigma(pJ_1 - J_2). \tag{2}$$

Now to find σ we recall that[1]

$$\xi = \frac{\partial W}{\partial p}, \qquad \pi = -\frac{\partial W}{\partial x} - p\frac{\partial W}{\partial y}. \tag{3}$$

As a consequence of (1), (2) and (3) we obtain the system of equations

$$(pJ_1 - J_2)\frac{\partial \sigma}{\partial p} + \left(p\frac{\partial J_1}{\partial p} - \frac{\partial J_2}{\partial p}\right)\sigma = 0,$$

$$(4) \quad (pJ_1 - J_2)\frac{\partial \sigma}{\partial x} + p(pJ_1 - J_2)\frac{\partial \sigma}{\partial y} + \left[\left(p\frac{\partial J_1}{\partial x} - \frac{\partial J_2}{\partial x}\right) + p\left(p\frac{\partial J_1}{\partial y} - \frac{\partial J_2}{\partial y}\right) + J_3\right]\sigma = 0.$$

This system may be written in the homogeneous form

[1] See Cohen, *An introduction to the Lie theory of one-parameter groups*, p. 186.

$$
\begin{aligned}
&A_1 f = \frac{\partial f}{\partial p} + M_1 \frac{\partial f}{\partial \sigma} = 0, \\
(5) \qquad & \\
&A_2 f = \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial y} + M_2 \frac{\partial f}{\partial \sigma} = 0,
\end{aligned}
$$

in which

$$(6) \quad M_1 = -\sigma \frac{p(\partial J_1/\partial p) - (\partial J_2/\partial p)}{pJ_1 - J_2},$$

$$(7) \quad M_2 = -\sigma \frac{(p(\partial J_1/\partial x) - \partial J_2/\partial x) + p(p(\partial J_1/\partial y) - \partial J_2/\partial y) + J_3}{pJ_1 - J_2}.$$

Adjoining to the system (5) the equations

$$(8) \quad A_3 f = (A_1 A_2) f = \frac{\partial f}{\partial y} + (A_1 M_2 - A_2 M_1) \frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial y} + M_3 \frac{\partial f}{\partial \sigma} = 0,$$

$$(9) \quad A_4 f = (A_1 A_3) f = (A_1 M_3 - A_3 M_1) \frac{\partial f}{\partial \sigma} = 0,$$

$$(10) \quad A_5 f = (A_2 A_3) f = (A_2 M_3 - A_3 M_2) \frac{\partial f}{\partial \sigma} = 0,$$

we see that the equations

$$(11) \qquad A_1 M_3 - A_3 M_1 = 0, \qquad A_2 M_3 - A_3 M_2 = 0,$$

are necessary and sufficient conditions in order that the system (4) have a solution. The system (5)–(8) implies the Jacobian complete system

$$
\begin{aligned}
&K_1 f = \frac{\partial f}{\partial x} + (M_2 - pM_3) \frac{\partial f}{\partial \sigma} = 0, \\
(12) \qquad &K_2 f = \frac{\partial f}{\partial y} + M_3 \frac{\partial f}{\partial \sigma} = 0, \\
&K_3 f = \frac{\partial f}{\partial p} + M_1 \frac{\partial f}{\partial \sigma} = 0.
\end{aligned}
$$

Either we may solve (12) or the equivalent total differential equation

$$(13) \qquad (M_2 - pM_3)dx + M_3 dy + M_1 dp - d\sigma = 0.$$

If $f = \psi(x, y, p, \sigma)$ is the solution of (12), then

$$\psi(x, y, p, \sigma) = c \tag{14}$$

will be the solution of (13), and conversely. Equation (14) determines σ in terms of x, y, and p. Since σ enters as a factor in M_1 and M_2, it is also a factor of M_3.[2] Hence, equation (13) can be written

$$d\sigma/\sigma = d\omega(x, y, p),$$

and so σ has the form

$$\sigma = ke^{\omega(x,y,p)}. \tag{15}$$

Several special formulas for σ may be found. For instance, if

$$M_1 = \phi_1(p)\sigma, \qquad M_2 = \phi_2(x)\sigma,$$

then $M_3 = 0$, and equation (13) reduces to

$$\phi_2(x)\sigma dx + \phi_1(p)\sigma dp - d\sigma = 0,$$

from which we obtain

$$\sigma = k \exp\left(\int \phi_1(p)dp + \int \phi_2(x)dx\right).$$

Therefore, the characteristic function takes the form

$$W = k(pJ_1 - J_2) \exp\left(\int \phi_1(p)dp + \int \phi_2(x)dx\right) \tag{16}$$

by virtue of (2). This special case will be of use in some examples to be considered later.

We summarize our results in the following theorem.

THEOREM. *The characteristic function W of the infinitesimal contact transformation leaving invariant a given differential equation $u = F(v)$ can be found by the formula*

$$W = k(pJ_1 - J_2)e^{\omega(x,y,p)}$$

if, and only if, the equations

$$A_1M_3 - A_3M_1 = 0, \qquad A_2M_3 - A_3M_2 = 0$$

are both satisfied for all values of x, y and p.

Now, to solve the differential equation $u = F(v)$ invariant under the known group

[2] If $M_1 = \sigma N_1$, $M_2 = \sigma N_2$, then $M_3 = A_1M_2 - A_2M_1 = \sigma(\partial N_2/\partial p - \partial N_1/\partial x - p\partial N_1/\partial y)$. This relation, together with (11), are the conditions in order that (13) be an exact differential when divided by σ.

$$(17)\qquad Bf = W_p\frac{\partial f}{\partial x} + (pW_p - W)\frac{\partial f}{\partial y} - (W_x + pW_y)\frac{\partial f}{\partial p},$$

we consider two cases:

(A) Both $\xi = W_p$ and $\eta = pW_p - W$ are free of p. This case occurs when W is linear in p. Then Bf represents an extended point transformation and the equation may be solved by introducing canonical variables.

(B) Either ξ or η, or both, contain p. Then Bf represents a general contact transformation.

In this case we may show that by a suitable change of variables the transformation reduces to a point transformation.[3] To this aim, let us define a finite contact transformation

$$(18)\qquad X = X(x, y, p), \qquad Y = Y(x, y, p), \qquad P = P(x, y, p)$$

in the following manner: $X = u$, $Y \neq X$ in involution with X, that is, such that $[XY] = 0$, or

$$(19)\qquad X_p\frac{\partial Y}{\partial x} + pX_p\frac{\partial Y}{\partial y} - (X_x + pX_y)\frac{\partial Y}{\partial p} = 0,$$

and P by the equation $P = Y_p/X_p$. The symbol for the transformed group will be

$$(20)\quad \overline{B}f = \bar{\xi}\frac{\partial f}{\partial X} + \bar{\eta}\frac{\partial f}{\partial Y} + \bar{\pi}\frac{\partial f}{\partial P} = BX\frac{\partial f}{\partial X} + BY\frac{\partial f}{\partial Y} + BP\frac{\partial f}{\partial P}\cdot$$

But $\bar{\xi} = BX = Bu = 0$ since u is invariant under Bf. Since $\bar{\xi} = \overline{W}_P$ this implies that $\overline{W}$ is free of P. Also, we find that $\bar{\eta}$ does not contain P because $\bar{\eta} = P\overline{W}_P - \overline{W} = -\overline{W}$. Hence, $\overline{B}f$ is an extension of the point transformation group

$$(21)\qquad Uf = -\overline{W}(X, Y)\frac{\partial f}{\partial Y}\cdot$$

This group can be reduced further by introducing the canonical variables

$$X^* = X, \qquad Y^* = -\int\frac{\partial Y}{\overline{W}(X, Y)}\cdot$$

Then the symbol of the infinitesimal transformation assumes the

[3] Cohen, loc. cit. p. 195, proves that the contact transformation reduces to a point transformation by assuming the corresponding differential equation solvable for p in the form $p = \omega(x, y)$.

simplest form

$$U^*f = \frac{\partial f}{\partial Y^*} \cdot$$

The equation $u = F(v)$ when written with the variables X, Y, P takes the form

$$\phi(X, Y, P) = 0. \tag{22}$$

This is also a differential equation, that is, $P = dY/dX$, since the relation $dY - PdX = \lambda(dy - pdx)$ which holds for any contact transformation implies $dY - PdX = 0$ whenever $dy - pdx = 0$. Since (22) will be invariant under (21) we are in position to solve (22), either directly or by introducing the canonical variables X^*, Y^* [this last step reduces the equation to the form $dY^*/dX^* = G(X^*)$]. Let

$$\psi(X, Y, c) = 0 \tag{23}$$

be the solution of (22). Passing back to the original variables we get a second differential equation

$$\Psi(x, y, p, c) = 0$$

which together with $u = F(v)$ determines the integral curves of the latter in terms of the parameter p.

Examples. I. Consider the differential equation

$$p + y/p = F(x + 2p). \tag{24}$$

Here $u = p + y/p$, $v = x + 2p$. Hence, it follows that $J_1 = 2/p$, $J_2 = 1 - y/p^2$, $J_3 = -1/p$, $pJ_1 - J_2 = 1 + y/p^2$, $M_1 = 2\sigma/p$, $M_2 = M_3 = 0$.

Formula (16) can be applied with $\phi_1(p) = 2/p$, $\phi_2(x) = 0$. Therefore, the characteristic function of the group is

$$W = k(1 + y/p^2)p^2 = k(p^2 + y).$$

Since a constant factor is irrelevant, we see that equation (24) is invariant under the infinitesimal contact transformation

$$Bf = 2p\frac{\partial f}{\partial x} + (p^2 - y)\frac{\partial f}{\partial y} - p\frac{\partial f}{\partial p} \cdot$$

By taking $X = v = x + 2p$ equation (19) reduces to

$$2\frac{\partial Y}{\partial x} + 2p\frac{\partial Y}{\partial y} - \frac{\partial Y}{\partial p} = 0. \tag{25}$$

The corresponding system of ordinary differential equations is

$$\frac{dx}{2} = \frac{dy}{2p} = \frac{dp}{-1} = \frac{dY}{0},$$

from which we obtain $Y = p^2 + y$ as a particular integral of (25). Finally we have $P = 2p/2 = p$. Introducing the new variables in (24) we get $dY/Y = dX/F(X)$. Hence, we have

$$Y = ce^{G(X)}, \qquad G(X) = \int \frac{dX}{F(X)}.$$

Passing back to the variables x, y, p we obtain

$$p^2 + y - ce^{G(x+2p)} = 0. \tag{26}$$

The system (24)–(26) furnishes the solution of the equation (24).

For instance, if $F(x+2p) = \tan(x+2p)$, equations (24) and (26) are respectively

$$p + y/p = \tan(x + 2p), \qquad p^2 + y = c \sin(x + 2p).$$

Solving for x and y we find

$$x = -2t + \arc\cos(t/c),$$
$$y = -t^2 \pm (c^2 - t^2)^{1/2},$$

which are the parametric equations of the solution, where $t = p$ is the parameter.

II. To apply the method to find the group leaving invariant some familiar types of ordinary differential equations, let us consider first the homogeneous equation

$$p = F(y/x).$$

We have $u = p$, $v = y/x$, $J_1 = -1/x$, $J_2 = -y/x^2$, $J_3 = 0$, $pJ_1 - J_2 = (y - px)/x^2$, $M_1 = 0$, $M_2 = 2\sigma/x$, $M_3 = 0$. By using formula (16) with $\phi_1(p) = 0$, $\phi_2(x) = 2/x$, we get (taking $k = -1$)

$$W = px - y.$$

Since W is linear in p we obtain the point transformation with symbol

$$Uf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y},$$

which corresponds to the so-called homotetic transformation.

For the linear equation $p + P(x)y = F(x)$ we have $u = p + Py$, $v = x$, $J_1 = 0$, $J_2 = 1$, $J_3 = -P$, $pJ_1 - J_2 = -1$, $M_1 = 0$, $M_2 = -\sigma P$, $M_3 = 0$.

By putting $\phi_1(p)=0$, $\phi_2(x)=-P$, $ke^c=1$ in formula (16) we obtain

$$W = -\exp\left(-\int Pdx\right).$$

Hence the symbol for the group has the form

$$Uf = \exp\left(-\int Pdx\right)\frac{\partial f}{\partial y}\cdot$$

III. Finally, we shall give a short table of some general types with the corresponding characteristic functions.[4]

Differential Equations	Characteristic Functions
$y = px + F[x\phi(p)]$	$kx\phi(p)$
$y = px + pF[y\phi(p)]$	$kyp\phi(p)$
$y + \phi(p) = pF\left[x + \int \phi'(p)dp/p\right]$	$k[y + \phi(p)]$
$e^x\phi(x + y + p) = F[e^x(p + 1)]$	$k\phi(x + y + p)$
$\dfrac{y + x\phi(p)}{p + \phi(p)} = xF\left[\log x + \int \dfrac{\phi'(p)dp}{p + \phi(p)}\right]$	$k[y + x\phi(p)]$.

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[4] I am indebted to my former students Miss C. Santana and Dr. R. Peña for the fourth and fifth types shown in the list.