A SUBSTITUTE FOR THE PICONE FORMULA

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It has been known for a number of years that the calculus of variations affords a powerful tool for the study of the oscillation of solutions of self-adjoint differential systems.[1] It is the purpose of the present paper to demonstrate how it can be used to replace and extend the Picone theorem.[2]

Consider the pair of self-adjoint differential equations

$$(1.1) \qquad \frac{d}{dx}(ru') + pu = 0,$$

$$(1.2) \qquad \frac{d}{dx}(r_1u') + p_1u = 0,$$

where, for definiteness, it is assumed that $r(x)$, $r_1(x)$, $r'(x)$, $r_1'(x)$, $p(x)$, $p_1(x)$ are continuous with r and r_1 positive on the interval $a \leq x \leq b$. With these equations we associate the functional identity

$$
(2) \qquad
\begin{aligned}
\int_a^b & [(r - r_1)u'^2 + (p_1 - p)u^2]dx \\
&= ruu' \Big|_a^b - \int_a^b u[(ru')' + pu]dx - \int_a^b [r_1u'^2 - p_1u^2]dx.
\end{aligned}
$$

The proof of (2) requires simply the integration by parts of the term ru'^2 in its left-hand member. In what follows it is convenient to *admit* to our discussion functions[3] $u(x)$ of class C' on the interval $a \leq x \leq b$ which vanish at a and b.

Repeated use is made of the following well known result.[4]

LEMMA. *If there exists an admissible curve $y = y(x)$ along which*

$$(3) \qquad J = \int_a^b (Ry'^2 - Py^2)dx < 0,$$

[1] See, for example, Morse [2, Chap. IV]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Cf. Bôcher [1, p. 54], Ince [1, p. 225].

[3] A function is said to be of class C' on an interval $a \leq x \leq b$ if it is continuous and has a continuous derivative on $a \leq x \leq b$.

[4] Cf. Morse [2, chap. 2].

a solution $\bar{y}(x) \not\equiv 0$ *of the differential system*

(4)
$$(Ry')' + Py = 0,$$
$$y(a) = 0$$

will have a zero on the interval $a < x < b$.

We return now to equations (1.1) and (1.2) and suppose that $u(x) \not\equiv 0$ is a solution of (1.1) having x_1 and x_2 as consecutive zeros $(x_1 < x_2)$. *We seek sufficient conditions that every solution* $u_1(x)$ *of* (1.2) *shall have at least one zero on the interval* $x_1 < x < x_2$. To that end we evaluate (2) with u, a, b replaced respectively by $u(x)$, x_1, x_2, obtaining the fundamental relationship

(5)
$$\int_{x_1}^{x_2} [(r_1 u'^2(x) - p_1 u^2(x)] dx$$
$$= -\int_{x_1}^{x_2} [(r - r_1)u'^2(x) + (p_1 - p)u^2(x)] dx,$$

which for simplicity we write as $I_1 = -I$. It is clear that if we prescribe conditions which will insure that I is positive, I_1 will then be negative. It will follow from the lemma that every solution $u_1(x)$ of (1.2) which vanishes at x_1 will vanish again on the interval $(x_1 < x < x_2)$. The elementary Sturm separation theorem then insures that every solution of (1.2) will then vanish at least once on the interval $x_1 < x < x_2$.

The Sturm-Picone (SP) conditions provide that $r \geq r_1$ and $p_1 \geq p$, with the latter inequality holding in the strict sense in at least one point of the interval of integration. Clearly these conditions will insure that $I > 0$, and hence that all solutions of (1.2) will have a zero on $x_1 < x < x_2$.

The SP conditions are clearly not very delicate. One does not need to require that the coefficients of u'^2 and u^2 be essentially positive to insure that I is positive. Indeed, we may apply the Lemma to I. We obtain at once the following result.

THEOREM.[5] *In equations* (1.1) *and* (1.2) *let* $r > r_1$ *and suppose that a solution* $u(x)$ *of* (1.1) *has* $x = x_1$ *and* $x = x_2$ *as consecutive zeros. If* $p(x)$ *and* $p_1(x)$ *are such that a solution* $w(x) \not\equiv 0$ *of the differential system*

(6.1)
$$\frac{d}{dx}[(r - r_1)w'] + (p - p_1)w = 0,$$

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[5] It should be pointed out that this result can be derived as a consequence of a theorem due to Morse [**2**, p. 102].

$$(6.2) \qquad\qquad w(x_1) = 0$$

*has the property that $w(x) \neq 0$ on the interval $x_1 < x \leq x_2$, every solution of
(1.2) has at least one zero on the interval $x_1 < x < x_2$. The conclusion of
this theorem will remain valid if $w(x) \neq 0$ on $x_1 < x < x_2$ and $w(x_2) = 0$,
provided the solution $u(x)$ does not satisfy equation* (6.1).[6]

The condition $r > r_1$ which is desirable in order that the functional
I shall be nonsingular may appear at first glance to be unduly restric-
tive. It is easily seen however that the theorem permits the study of
the oscillation of solutions of (1.1) and (1.2) even when $r \leq r_1$. We may
simply replace $r(x)$ and $p(x)$ by $kr(x)$ and $kp(x)$, respectively, where k
is any positive constant such that $kr(x) > r_1(x)$. Such a determination
of k is possible since $r(x)$ is positive throughout its domain of defini-
tion.

The observation above suggests incidentally the following simple
extension of the SP theorem.

CoROLLARY. *Let $u(x) \neq 0$ be a solution of* (1.1) *which has zeros at
$x = x_1$ and $x = x_2$. If there exists a positive constant k such that both the
conditions*

$$(7) \qquad\qquad kr - r_1 \geq 0, \qquad p_1 - kp \geq 0$$

hold, every solution $u_1(x)$ of (1.2) *has a zero on the interval $x_1 < x < x_2$,
provided that at some point of this interval the latter inequality holds in
the strict sense.*

The following example demonstrates that the theorem includes
cases not covered by the SP theory even in the extended form sug-
gested by the corollary.

EXAMPLE. The Legendre equation

$$\frac{d}{dx}\left[(1 - x^2)y'\right] + 12y = 0$$

has the solution $y = P_3(x) = (5x^3 - 3x)/2$ which vanishes at $x = 0$ and at
$x = (3/5)^{1/2}$. We shall show that every solution of the differential
equation

$$\frac{d}{dx}\left[(3/4 - x^2)y'\right] + 8y = 0$$

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[6] A complete bibliography of oscillation and comparison theorems for Sturm-
Liouville systems cannot be given here. Closely related papers are Bliss and Schoen-
berg [1] and Morse [1] to which the reader's attention is directed.

has a zero on the interval $0 < x < (3/5)^{1/2}$. Here the analogues of equations (6.1) and (6.2) are

$$\frac{d}{dx}(w'/4) + 4w = 0, \qquad w(0) = 0.$$

It is clear that $w(x)$ may be taken as $\sin 4x$ which is positive on the interval $0 < x < (3/5)^{1/2}$. An appeal to the theorem completes the proof. It is readily verified that the SP theory does not apply even in the extended form suggested by the corollary.

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