and hence (6) holds for every sufficiently large fixed k. Now put in (5) $n=N_\nu=n_\nu-a_1a_2\cdots a_t$. Then for every fixed M

$$1 \geqq r_0u_{N_\nu}+r_1u_{N_\nu-1}+\cdots+r_Mu_{N_\nu-M}.$$

As $\nu\to\infty$ all terms $u_{N_\nu-k}\to\lambda$ and hence

$$1 \geqq \lambda(r_0+r_1+\cdots+r_M)$$

or $\lambda\leqq 1/m$ (with $\lambda=0$ if $\sum r_n=\infty$).

If $m<\infty$ we can use a similar argument for $\mu=\lim\inf u_n$ to show that $\mu\geqq 1/m$. This proves the theorem.

SYRACUSE UNIVERSITY AND
CORNELL UNIVERSITY

A CONSISTENCY THEOREM

H. D. BRUNK[1]

1. Introduction. Of primary importance in a theory of representation of functions by series which do not necessarily converge is its consistency theorem, which states that if a series which represents a function F converges to a function Φ, then $F\equiv\Phi$. Such a theorem for asymptotic representation in a strip region of a function by Dirichlet series with a certain logarithmic precision, an idea introduced by Mandelbrojt [1],[2] is the subject matter of this note. From it follow similar theorems for less general extensions of the idea of asymptotic series. The method consists in using the proof of the fundamental theorem in [1] to set up a homogeneous linear differential equation of infinite order with constant coefficients, which must be satisfied by the difference $F-\Phi$; then applying a method of Ritt to show that the only solution is identically zero.

The notation used by Mandelbrojt in [1] will be used here also. Let $\{\lambda_n\}$ be an increasing sequence of positive numbers $(0<\lambda_n\uparrow)$. Denote by $N(\lambda)$, defined for $\lambda>0$, the *distribution function* of $\{\lambda_n\}$; that is, the number of terms in the sequence $\{\lambda_n\}$ less than λ; and by $D(\lambda)$ the *density function* of $\{\lambda_n\}$: $D(\lambda)=N(\lambda)/\lambda$. Let $D^\cdot$ represent the *upper density*: $D^\cdot=\lim\sup_{\lambda\to\infty}D(\lambda)$; and $D^\cdot(\lambda)$ the *upper density function* of $\{\lambda_n\}$: $D^\cdot(\lambda)=\text{l.u.b.}_{x\geqq\lambda}D(x)$; clearly $D^\cdot(\lambda)$ is continuous and decreases to $D^\cdot$ (unless $D^\cdot(\lambda)\equiv D^\cdot=\infty$).

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[1] The author is indebted to Professor Mandelbrojt for suggesting the problem considered in this note.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

The *mean density function* is $\overline{D}(\lambda)$: $\overline{D}(\lambda)=[\int_0^\lambda D(x)dx]/\lambda$; $\overline{D}^\cdot$, the *upper mean density*: $\overline{D}^\cdot=\lim\sup_{\lambda\to\infty}\overline{D}(\lambda)$; and $\overline{D}^\cdot(\lambda)=\text{l.u.b.}_{x\geqq\lambda}\overline{D}(x)$, the *upper mean density function*. The function $\overline{D}^\cdot(\lambda)$ decreases to $\overline{D}^\cdot$ as λ becomes infinite. Also $\overline{D}^\cdot\leqq D^\cdot$ (cf. [1]); and if either $\overline{D}^\cdot$ or $D^\cdot$ is finite, so is the other [1, p. 353].

DEFINITION. Let Δ be a region of the s-plane ($s=\sigma+it$) containing points with arbitrarily large real part, and $F(s)$ a function holomorphic in Δ. Let n be a positive integer, $\{d_k\}$ a sequence of complex numbers, and $p_n(x)$ a function increasing to infinity such that for s in Δ and for x sufficiently large

$$\text{(1)}\qquad \underset{m\geqq n}{\text{g.l.b.}}\ \underset{\sigma\geqq x}{\text{l.u.b.}}\left|F(s)-\sum_{k=1}^{m}d_ke^{-\lambda_k s}\right|\leqq e^{-p_n(x)}.$$

The sums $\sum_{k=1}^m d_ke^{-\lambda_k s}$ ($m\geqq n$) *are then said to represent* $F(s)$ *in* Δ *with the logarithmic precision* $p_n(\sigma)$. If the series $\sum_{k=1}^\infty d_ke^{-\lambda_k s}$ converges to $F(s)$ for σ sufficiently large, the logarithmic precision is identically infinite for each n [1, p. 356].

THEOREM I. *If*

(i) *the sequence* $\{\lambda_n\}$ *has finite upper mean density* $\overline{D}^\cdot$;

(ii) *the function* $F(s)$ *is holomorphic in the region* Δ *of the s-plane* ($s=\sigma+it$) *given by*: $\{\sigma>b,\ |t|<\pi g(\sigma)\}$, *where b is a real constant, and* $g(\sigma)$ *is a continuous function of bounded variation defined for* $\sigma>b$ *such that* $\lim_{\sigma\to\infty}g(\sigma)>\overline{D}^\cdot$;

(iii) $\text{g.l.b.}_{n\geqq 1}(\lambda_{n+1}-\lambda_n)>0$;

(iv) *for infinitely many positive integers n, the sums* $\sum_{k=1}^m d_ke^{-\lambda_k s}$ ($m\geqq n$) *represent* $F(s)$ *in* Δ *with a logarithmic precision* $p_n(\sigma)$ *satisfying*[3]

$$\text{(2)}\qquad \int^{\sigma} p_n(\sigma)\exp\left(-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-\overline{D}^\cdot(p_n(u))}\,d\sigma\right)=\infty;$$

then the series $\sum_{k=1}^\infty d_ke^{-\lambda_k s}$ *converges to* $F(s)$ *for* σ *sufficiently large*; *that is, it has a half-plane of convergence, in which it converges to* $F(s)$.

Theorem I gives immediately a corresponding result for the theory of asymptotic representation with respect to a sequence of functions[4]

[3] The omission of the lower limits on the integrals indicates that the relation is to hold provided these limits are chosen sufficiently large.

[4] The following definition of asymptotic representation with respect to an asymptotic sequence, as well as the discussion of its relation to representation with a certain logarithmic precision which appears in §3, is based on lectures given at The Rice Institute in 1947 by Professor Mandelbrojt.

$\{A_n(\sigma)\}$. Consider a sequence of continuous functions $\{A_n(\sigma)\}$ having the property that

(3) $A_n(\sigma)$ decreases to zero as $\sigma \to \infty$ $(n \geqq 1)$;

(4) $A_{n+1}(\sigma) = O(A_n(\sigma))$ as $\sigma \to \infty$ $(n \geqq 1)$.

DEFINITION. The function $F(s)$ is represented asymptotically in a region Δ containing points with arbitrarily large real part by $\sum_{k=1}^{\infty} d_k e^{-\lambda_k s}$ with respect to the *asymptotic sequence* $\{A_n(\sigma)\}$ if there exists a real number σ_0 independent of n such that

$$\left| F(s) - \sum_{k=1}^{n} d_k e^{-\lambda_k s} \right| < A_n(\sigma) \quad \text{for} \quad \sigma > \sigma_0 \qquad (n \geqq 1). \tag{5}$$

Define $A(\sigma)$, the *lower envelope* of the sequence $\{A_n(\sigma)\}$, as $\text{g.l.b.}_{n \geqq 1} A_n(\sigma)$, and set $p(\sigma) = -\log A(\sigma)$. The following theorem is a corollary of Theorem I.

THEOREM II. *If* (i), (ii), (iii), *and*

(iv′) *$F(s)$ is represented asymptotically in Δ by $\sum d_k e^{-\lambda_k s}$ with respect to an asymptotic sequence $\{A_n(\sigma)\}$ such that*

$$\int^{\infty} p(\sigma) \exp\left(- \frac{1}{2} \int^{\sigma} \frac{du}{g(u) - \overline{D}^{\cdot}(p(u))} \right) d\sigma = \infty \tag{2′}$$

then the series converges to $F(s)$ for σ sufficiently large.

In both theorems the upper mean density $\overline{D}^{\cdot}$ may be replaced by the upper density $D^{\cdot}$, the upper mean density function $\overline{D}^{\cdot}(\lambda)$ being replaced by the upper density function $D^{\cdot}(\lambda)$. Moreover (2) and (2′) may be replaced by hypotheses corresponding to those in forms C, D, and E of the statement of the fundamental theorem in [1]. In particular (2) may be replaced by:

(2a) There exist continuous functions $h_n(\sigma)$ decreasing to $\overline{D}^{\cdot}$, and increasing functions $C_n(\sigma)$ such that

$$2\nu(h_n(\sigma)) - p_n(\sigma) < - C_n(\sigma),$$

$$(n \geqq 1)$$

$$\int^{\infty} C_n(\sigma) \exp\left(- \frac{1}{2} \int^{\sigma} \frac{du}{g(u) - h_n(u)} \right) d\sigma = \infty;$$

where $\nu(D) = \text{l.u.b.}_{\lambda > 0}\, \lambda(\overline{D}(\lambda) - D) = \text{l.u.b.}_{\lambda > 0} \int_0^{\lambda} [D(x) - D] dx$. This is a non-negative, nonincreasing continuous function of D for $D > \overline{D}^{\cdot}$, *called the excess function of the sequence* $\{\lambda_n\}$.

If the $\{A_n(\sigma)\}$ have the special form $A_n(\sigma) = M_n e^{-\lambda_n \sigma}$, Theorem II reduces to a theorem proved by Mandelbrojt by a different method, mentioned briefly in §4.

2. **Proof of I.** The proof of Theorem I falls naturally into two parts, Lemmas 1 and 2 below.

LEMMA 1. *Under the hypotheses of Theorem* I, *the series* $\sum d_k e^{-\lambda_k s}$ *converges for* σ *sufficiently large.*

This lemma was proved by Mandelbrojt in his lectures at The Rice Institute in 1947.

One may assume that $p_n(\sigma)$ is a decreasing function of n for each σ. For if (1) is satisfied with right-hand member $e^{-p_n^*(x)}$, it is also satisfied if $p_n^*(x)$ is replaced by $p_n(x) = \text{l.u.b.}_{j\geqq n} p_j^*(x)$, since for each $j \geqq n$ (writing $\phi_m(x) = \text{l.u.b.}_{\sigma \geqq x} \left| F(s) - \sum_{k=1}^{m} d_k e^{-\lambda_k s} \right|$) $\text{g.l.b.}_{m\geqq n}\phi_m(x) \leqq \text{g.l.b.}_{m \geqq j}\phi_m(x) \leqq e^{-p_j^*(x)}$ hence $\text{g.l.b.}_{m\geqq n}\phi_m(x) \leqq \text{g.l.b.}_{j\geqq n} e^{-p_j^*(x)} = e^{-p_n(x)}$. Moreover, if the $p_n^*(\sigma)$ satisfy (2) (or 2a) so do the $p_n(\sigma)$.

Hence if (2) is satisfied for any integer n, it is satisfied also for any smaller integer ($\overline{D}^{\cdot}(\lambda)$ is a decreasing function of λ). Thus (iv) is satisfied for all $n \geqq 1$. The hypotheses of Theorem I are then sufficient in order that the conclusion of Mandelbrojt's fundamental theorem [1, p. 357] may hold for $n \geqq 1$:

$$|d_n| \leqq A\lambda_n \Lambda_n^* e^{\lambda_n \sigma_0} \qquad \text{for some } \sigma_0 > b;$$

where A is independent of n, depending on the sequence $\{\lambda_n\}$ and an upper bound on F in Δ (it is clear from (1) that F is bounded in Δ), and where

$$\Lambda_n^* = \frac{1}{|\Lambda_n(i\lambda_n)|}, \qquad \Lambda_n(iz) = \prod_{k \neq n}\left(1 - \frac{z^2}{\lambda_k^2}\right).$$

It follows from Mandelbrojt's inequalities on Λ_n^* [1, p. 353] that if $\lim\inf_{n\to\infty}(\lambda_{n+1} - \lambda_n) = h > 0$, there exists a number $B = B(h, D^{\cdot})$ such that $\log \Lambda_n^* \leqq B\lambda_n$ for n sufficiently large. Hence

$$\begin{aligned} &\log |d_n| \leqq \log(A\lambda_n) + B\lambda_n + \lambda_n\sigma_0, \\ &(\log |d_n|)/\lambda_n \leqq B + \sigma_0 + (\log(A\lambda_n))/\lambda_n. \end{aligned} \tag{6}$$

But since the upper density $D^{\cdot}$ is finite, a necessary and sufficient condition that $\sum d_k e^{-\lambda_k s}$ converge for σ sufficiently large is that $\lim\sup_{n\to\infty}(\log |d_n|)/\lambda_n$ be finite (not positively infinite), and indeed this number is the abscissa of convergence.[5] This completes the proof of Lemma 1.

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[5] Cf. V. Bernstein, *Series de Dirichlet*, Paris, Gauthier-Villars, 1933, p. 4.

LEMMA 2. *If* (i), (ii), (iv), *and*
(v) $\sum d_k e^{-\lambda_k s}$ *converges for* σ *sufficiently large, then it converges to* $F(s)$.

Let $\Phi(s)$ denote the function to which the series converges under (v). Define the coefficients $\{c_k^{(n)}\}$ by

$$\Lambda_n(iz) = \prod_{k\neq n}\left(1 - \frac{z^2}{\lambda_k^2}\right) = \sum_{k=0}^{\infty}(-1)^k c_k^{(n)} z^{2k}. \tag{7}$$

In the proof of his fundamental theorem, Mandelbrojt proves that in a certain closed strip $\bar{\Delta}$ (*not* the closure of Δ) given by $\{\sigma \geqq d, |t| \leqq G(\sigma)\}$ (where d is a certain real number and $G(\sigma)$ a positive function, both independent of n), the series $\sum_{k=0}^{\infty}(-1)^k c_k^{(n)} F^{(2k)}(s)$ converges absolutely and uniformly to the function $d_n\Lambda_n(i\lambda_n)e^{-\lambda_n s}$. Since $\Phi(s)$ satisfies the same hypotheses, $\sum(-1)^k c_k^{(n)}\Phi^{(2k)}(s)$ converges absolutely and uniformly in $\bar{\Delta}$ to the same function; and if $H(s) = F(s) - \Phi(s)$, then for s in $\bar{\Delta}$ and for $n \geqq 1$,

$$\sum(-1)^k c_k^{(n)} H^{(2k)}(s) \equiv 0. \tag{8}$$

From the definition (7) of the $c_k^{(n)}$ it is seen that (8) may be written symbolically as

$$[(1 - D^2/\lambda_1^2)(1 - D^2/\lambda_2^2)\cdots \cdot(1 - D^2/\lambda_{n-1}^2)(1 - D^2/\lambda_{n+1}^2)\cdots]H(s) \equiv 0,$$

where D indicates differentiation with respect to s. A method due to Ritt[6] may then be used to show that $H(s) \equiv 0$, which is the desired result. Define functions $N_m(z)$ and constants $c_{k,m}$ by:

$$N_m(z) = \prod_{k>m}\left(1 - \frac{z^2}{\lambda_k^2}\right) = \sum_{k=0}^{\infty}(-1)^k c_{k,m} z^{2k} \qquad (m \geqq 1). \tag{9}$$

Then for each m,

$$c_{0,m} = 1; \qquad \text{and for each } k, \tag{10}$$
$$c_{k,m} \text{ decreases to zero as } m \to \infty.$$

For

$$c_{k,m} = \sum_{i_1=m+1}^{\infty}\frac{1}{\lambda_{i_1}^2}\sum_{i_k>i_{k-1}>\cdots>i_2>i_1}\frac{1}{\lambda_{i_2}^2}\cdot\frac{1}{\lambda_{i_3}^2}\cdots\frac{1}{\lambda_{i_k}^2} \qquad (k \geqq 1, m \geqq 1),$$

[6] Ritt, J. F., *On a general class of linear homogeneous differential equations of infinite order with constant coefficients*, Trans. Amer. Math. Soc. vol. 18 (1917) pp. 27–49; cf. in particular Theorem XIV.

the second sum being bounded by a number, $c_{k-1}^{(1)}$, independent of m:

$$c_{k-1}^{(1)} = \sum_{i_k > i_{k-1} > \cdots > i_2 \geqq 1} \frac{1}{\lambda_{i_2}^2} \cdot \frac{1}{\lambda_{i_3}^2} \cdots \frac{1}{\lambda_{i_k}^2};$$

from which it is apparent also that $c_{k,m} \leqq c_k^{(1)}$ ($m \geqq 1$, $k \geqq 0$). Since $\sum(-1)^k c_k^{(1)} H^{(2k)}(s)$ converges absolutely and uniformly in $\overline{\Delta}$, the series $\sum(-1)^k c_{k,m} H^{(2k)}(s)$ also converges absolutely and uniformly in $\overline{\Delta}$ to a function $h_m(s)$ holomorphic in $\overline{\Delta}$. For each s in $\overline{\Delta}$ the convergence is uniform with respect to m also, so that $\lim_{m\to\infty} h_m(s) = H(s)$ by (10). Symbolically

$$h_m(s) = [(1 - D^2/\lambda_{m+1}^2)(1 - D^2/\lambda_{m+2}^2) \cdots]H(s). \tag{11}$$

By (8) and the definition of the $h_m(s)$, for each $m \geqq 1$,

$$\begin{aligned} &(1 - D^2/\lambda_2^2)(1 - D^2/\lambda_3^2) \cdots (1 - D^2/\lambda_m^2) \quad h_m(s) \equiv 0, \\ &(1 - D^2/\lambda_1^2)(1 - D^2/\lambda_3^2) \cdots (1 - D^2/\lambda_m^2) \quad h_m(s) \equiv 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &(1 - D^2/\lambda_1^2)(1 - D^2/\lambda_2^2) \cdots (1 - D^2/\lambda_{m-1}^2) h_m(s) \equiv 0. \end{aligned} \tag{12}$$

The operator $\prod_{j=1(j\neq n)}^{j=m}(1-D^2/\lambda_j^2)$ is evidently equivalent to $\sum_{j=1}^{m-1}(-1)^j a_{j,m}^{(n)} D^{2j}$, the $a_{j,m}^{(n)}$ being the coefficients in the Taylor's expansion of $\prod_{j=1(j\neq n)}^{m}(1-z^2/\lambda_j^2)$. Since we have $(\sum_{j=1}^{m-1}(-1)^j a_{j,m}^{(n)} z^{2j}) \cdot (\sum_{k=0}^{\infty}(-1)^k c_{k,m} z^{2k}) \equiv \sum_{u=0}^{\infty}(-1)^u c_u^{(n)} z^{2u}$, one easily verifies that $\sum_{j=1}^{m-1}(-1)^j a_{j,m}^{(n)} D^{2j}(\sum_{k=0}^{\infty}(-1)^k c_{k,m} D^{2k} H(s)) \equiv \sum_{u=0}^{\infty}(-1)^u c_u^{(n)} D^{2u} H(s) \equiv 0$, these being equations (12).

This is a system of m differential equations of finite order with constant coefficients. In order to satisfy the first of these equations, $h_m(s)$ must be a linear combination of powers $\pm\lambda_2 s$, $\pm\lambda_3 s$, $\pm\lambda_4 s, \cdots$, $\pm\lambda_m s$ of e; to satisfy the second, a linear combination of powers $\pm\lambda_1 s$, $\pm\lambda_3 s, \cdots, \pm\lambda_m s$, and so on. It follows that $h_m(s) \equiv 0$ ($m \geqq 1$). Since $H(s) = \lim_{m\to\infty} h_m(s)$, also $H(s) \equiv 0$, which completes the proof of the lemma; and with Lemma 1, completes the proof of Theorem 1.

3. **Proof of II.** In his lectures at The Rice Institute referred to above, Mandelbrojt proved the following lemma.

LEMMA 3. *Representation with respect to an asymptotic sequence $\{A_n(\sigma)\}$ implies representation for each positive integer n with logarithmic precision $p(\sigma) - K_n$ for certain positive constants K_n, where $p(\sigma) = -\log A(\sigma)$, and where $A(\sigma)$ is the lower envelope of $\{A_n(\sigma)\}$: $A(\sigma) = \text{g.l.b.}_{n\geqq 1} A_n(\sigma)$.*

By definition (3)–(5), $\text{l.u.b.}_{\sigma\geqq x}\left|F(s)-\sum_1^m d_k e^{-\lambda_k s}\right|\leqq A_m(x)$; and there exist constants $B_2, B_3, \cdots$ such that for $x>\sigma_0$ the inequalities $A_2(x)<B_2A_1(x), A_3(x)<B_3A_2(x), \cdots, A_n(x)<B_nA_{n-1}(x)$ hold simultaneously. Hence $A_n(x)<B_nA_{n-1}(x)<B_nB_{n-1}A_{n-2}(x)<\cdots<B_nB_{n-1}\cdots B_2A_1(x)$. Choose R_n greater than 1 so as to be greater than $\max_{0\leqq j\leqq n-2}\prod_{i=0}^{j}B_{n-i}$ also. Then for each fixed n, $\text{g.l.b.}_{m\geqq n}R_nA_m(x)>\text{g.l.b.}_{m\geqq n}A_m(x)$, and $\text{g.l.b.}_{m<n}R_nA_m(x)>A_n(x)$; hence we have $\text{g.l.b.}_{m\geqq 1}R_nA_m(x)>\text{g.l.b.}_{m\geqq n}A_m(x)$. Thus

$$\begin{aligned}\underset{m\geqq n}{\text{g.l.b.}}\ \underset{\sigma\geqq x}{\text{l.u.b.}}\left|F(s)-\sum_1^m d_k e^{-\lambda_k s}\right| &\leqq \underset{m\geqq n}{\text{g.l.b.}}\ A_m(x)\\ &< \underset{m\geqq 1}{\text{g.l.b.}}\ R_nA_m(x) = R_nA(x) = \exp\left(\log A(x)+K_n\right)\\ &= \exp-\left[p(x)-K_n\right] \qquad\qquad \text{for } x>\sigma_0,\end{aligned}$$

where $K_n=\log R_n>0$, from which the lemma follows.

It will be sufficient, then, to show that hypothesis (iv′) of Theorem II implies (iv) of Theorem I, with (2a) replacing (2). Mandelbrojt shows in [1] that (iv′) implies the existence of a function $h(\sigma)$, continuous and decreasing to $\overline{D}^{\cdot}$, and an increasing function $C(\sigma)$ such that

$$2\nu(h(\sigma))-p(\sigma)<-C(\sigma),$$

$$\int^{\infty}C(\sigma)\exp\left(-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-h(u)}\right)d\sigma=\infty.$$

Put $p_n(\sigma)=p(\sigma)-K_n$, $h_n(\sigma)=h(\sigma)$, and $C_n(\sigma)=C(\sigma)-K_n$ for $n\geqq 1$. Since $\lim_{u\to\infty}g(u)>\overline{D}^{\cdot}$, the denominator in the integral appearing as a power of e is bounded above and below by positive numbers for u sufficiently large. Hence $\int^\infty K_n\{\exp\left[2^{-1}\int^\sigma du/(g(u)-h(u))\right]\}d\sigma<\infty$ for $n\geqq 1$, and

$$2\nu(h_n(\sigma))-p_n(\sigma)<-C_n(\sigma),$$

$$\text{(2a)}\qquad \int^{\infty}C_n(\sigma)\exp\left(-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-h(u)}\right)d\sigma=\infty$$

holds. The hypotheses of Theorem I are then satisfied, and the conclusion, that $\sum d_k e^{-\lambda_k s}$ converges to $F(s)$ for σ sufficiently large, follows.

4. **Remarks.** Mandelbrojt has given a simpler and more direct proof of Lemma 2 for the case in which $A_n(\sigma)=M_ne^{-\lambda_n\sigma}$, under less restrictive conditions: those obtained in removing the restriction

$\lim g(\sigma) > \overline{D}^{\cdot}$ (still assuming $\lim g(\sigma) > 0$) and in replacing $g(u) - \overline{D}^{\cdot}(p(u))$ in (2′) by $g(u)$. This proof depends on writing

$$\begin{aligned} |H(s)| = |F(s) - \Phi(s)| &= \left| F(s) - \sum_1^n d_k e^{-\lambda_k s} - \sum_{n+1}^{\infty} d_k e^{-\lambda_k s} \right| \\ &\leqq M_n e^{-\lambda_n \sigma} + \left| \sum_{n+1}^{\infty} d_k e^{-\lambda_k s} \right|, \end{aligned}$$

and considering separately the two possibilities: we have either $\liminf_{n\to\infty} (\log M_n)/\lambda_n < \infty$, or $\lim_{n\to\infty} (\log M_n)/\lambda_n = \infty$. If $B_1(\sigma) = \text{g.l.b.}_{n\geqq 1}[M_n e^{-\lambda_n\sigma} + |\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}|]$, and $B_1^*(\sigma) = \text{g.l.b.}_{n\geqq 1} M_n e^{-\lambda_n \sigma}$, it may be shown that, in either case, if $B_1^*(\sigma)$ satisfies the condition

$$\int^{\infty} -\log B_1^*(\sigma) \exp\left[-\frac{1}{2} \int^{\sigma} \frac{du}{g(u)} \right] d\sigma = \infty \tag{13}$$

then $B_1(\sigma)$ does also. For by Theorem II (Lemma 1) if (13) is satisfied the series $\sum_1^{\infty} d_k e^{-\lambda_k s}$ converges for σ sufficiently large. One then sees that if $s = a$ (a real) is a point of convergence of the series, then $|\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}| = |\sum_{n+1}^{\infty} d_k e^{-\lambda_k(s-a)} e^{-\lambda_k a}| \leqq M e^{-\lambda_k(\sigma - a)}$ where $M = \sum_1^{\infty} |d_k| e^{-\lambda_k a}$. Then if $\liminf_{n\to\infty} (\log M_n)/\lambda_n < \infty$, $B_1(\sigma) \equiv 0$; while if $\lim_{n\to\infty} (\log M_n)/\lambda_n = \infty$, then $M_n > M e^{\lambda_n a}$ for n sufficiently large, and $|\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}| < M_n e^{-\lambda_n \sigma}$, hence $B_1(\sigma) < 2B_1^*(\sigma)$. It follows that $H(s) \equiv 0$, by a theorem of Mandelbrojt and MacLane ([**2**, Theorem I]; cf. also [**1**, Lemma 1, p. 360], obtainable from [**2**, Theorem **1**], by the obvious transformation).

Similar proofs of Lemma 2 may be given provided the $\{A_n(\sigma)\}$ satisfy any of a number of conditions which may easily be formulated: in particular if $A^*(\sigma)$ satisfies (13), where $A^*(\sigma) = \liminf_{n\to\infty} A_n(\sigma)$, or, in general, if $B(\sigma)$ satisfies (13), where $B(\sigma) = \text{g.l.b.}_{n\geqq 1} \{ |\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}| + A_n(\sigma) \}$. Professor Mandelbrojt has remarked to the author that it follows from (6) that

$$\sum |d_k| e^{-\lambda_k a} < A \sum \lambda_k e^{-\lambda_k(a - B - \sigma_0)} < K_\epsilon \sum e^{-\lambda_k(a - B - \sigma_0 - \epsilon)}$$

for each positive number ϵ, K_ϵ being suitably chosen. Thus for $a > B + \sigma_0$, there exists a positive number $K = K(a)$ such that $\sum |d_k| e^{-\lambda_k a} < K$ (since $\sum e^{-\lambda_k s}$ converges for $\sigma > 0$), and such that $\sum_{n+1}^{\infty} |d_k| e^{-\lambda_k \sigma} = \sum_{n+1}^{\infty} |d_k| e^{-\lambda_k(\sigma - a)} e^{-\lambda_k a} \leqq K e^{-\lambda_n(\sigma - a)}$. Thus the conclusion of Lemma 2 can be shown to hold if $B^*(\sigma)$ satisfies (13), where $B^*(\sigma) = \text{g.l.b.}_{n\geqq 1}(K e^{-\lambda_n(\sigma - a)} + A_n(\sigma))$.

It may be remarked that a theorem of Mandelbrojt and MacLane ([**2**, Theorem II], a suitable transformation being made if $0 < \lim g(\sigma) \neq \pi/2$) constitutes a converse of Lemma 2 for asymptotic sequences, $g(u)-\overline{D}^{\cdot}(p(u))$ in (2′) being replaced by $g(u)$, as it may be under conditions on the $A_n(\sigma)$ discussed above. For by their theorem, given

$$\int^{\infty} p(\sigma) \exp\left[-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)}\right]d\sigma < \infty,$$

there exists a function $F(s)$ holomorphic in Δ, not identically zero, such that $\left|F(s)-\sum_1^n 0e^{-\lambda_k s}\right| < e^{-p(\sigma)}$, hence $\left|F(s)-\sum_1^n 0e^{-\lambda_k s}\right| < A_n(\sigma)$ if $\{A_n(\sigma)\}$ is any asymptotic sequence with g.l.b.$_{n\geqq 1}$ $A_n(\sigma)=A(\sigma)=e^{-p(\sigma)}$; so that $F(s)$ is represented asymptotically in Δ by the series $\sum d_k e^{-\lambda_k s}$ with $d_k=0$ $(k\geqq 1)$ with respect to the asymptotic sequence $\{A_n(\sigma)\}$, without being identically zero.

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R. D. Carmichael, *On Euler's ϕ function*, vol. 13, pp. 241–243; vol. 54, p. 1192.

Vol. 54, p. 1192, lines 2 and 9. For "Hedburg" read "Hedberg."

Vol. 54, p. 1192, line 10. For "$2^{28}+1$ and $2^{29}+1$" read "$2^{2^8}+1$ and $2^{2^9}+1$."