ON THE MEAN MODULUS OF AN ANALYTIC FUNCTION

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Throughout this paper $f=f(z)$ will denote an analytic function of the complex variable z in the open unit circle $|z|<1$. The circle $C(r)$, on which $|z|=r$, of radius $r\geqq 0$ about the origin $z=0$ lies in the region of analyticity of f provided $r<1$. For every positive real parameter t $(0<t<\infty)$ the mean of order t of the modulus of f on the circle $C(r)$ is defined as

$$(1)\qquad M_t(r;f)=\left[\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^t d\theta\right]^{1/t}.$$

For fixed f and r this mean modulus $M_t(r;f)$ as a function of t is continuous, nonnegative, nondecreasing, and is bounded above by the maximum modulus of f on $C(r)$ [1, 2].[1] Therefore the limit of $M_t(r;f)$ exists as $t\to 0$ and $t\to\infty$. This limit is defined to be the mean modulus of f on $C(r)$ of order 0 and of order ∞ respectively. It may be shown that the mean modulus of order 0 is the geometric mean of the modulus of f on $C(r)$, which is simply evaluated by Jensen's formula, and that the mean modulus of order ∞ is the maximum modulus of f on $C(r)$ [1, 2]. Thus $M_t(r;f)$ is defined for all parameters t in the compact infinite interval $0\leqq t\leqq\infty$.

For fixed f and t $(0\leqq t\leqq\infty)$ the mean modulus $M_t(r;f)$ as a function of r in the interval $0\leqq r<1$ is continuous, nonnegative, nondecreasing, and, except for the limiting parameters 0 and ∞, possesses a continuous derivative with respect to r [1, 3]. Moreover, its logarithm is a nondecreasing convex function of $\log r$ (for $t=\infty$ this is the Hadamard three-circle theorem) [1, 3].

We shall be concerned here with the convexity of the mean modulus $M_t(r;f)$ as a function of r. Let $T(f)$ be the set of all parameters t in the compact infinite interval $0\leqq t\leqq\infty$ such that $M_t(r;f)$ is a convex function of r in the interval $0\leqq r<1$. Since $M_t(r;f)$ is continuous with respect to the parameter t and since any function which is the limit of convex functions is also convex, the set $T(f)$ is a closed and hence compact subset of the parameter interval $0\leqq t\leqq\infty$. The set $T(f)$ need not, however, coincide with the entire parameter interval and indeed

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[1] Numbers in brackets refer to the references cited at the end of the paper.

may be a bounded subset of this interval as shown by the following example. Let

$$f(z) = (z + \epsilon)/(1 + \epsilon z), \qquad |z| < 1, 0 < \epsilon < 1.$$

This linear fractional function maps the open unit circle onto itself and maps the circle $C(r)$ into a circle whose maximal distance from the origin is attained at the image point $f(r)$. Therefore

$$M_\infty(r; f) = f(r),$$

which is easily seen to be a strictly concave function of r. Thus the set $T(f)$ for this function f does not contain the parameter ∞ and hence, being closed, is bounded.

THEOREM. *The set $T(f)$ contains the values $t = 2/k$ ($k = 1, 2, 3, \cdots$) and their limit value $t = 0$. Furthermore, if f has at most k zeros counting multiplicities ($k = 1, 2, 3, \cdots$), then $T(f)$ contains the closed interval $0 \leqq t \leqq 2/k$.*

PROOF. Evidently the theorem is true for a constant function f, so we may assume that f is nonconstant. We shall first investigate the convexity of the mean modulus $M = M_t(r; f)$ in any open interval $\alpha < r < \beta$ such that the associated open annulus $\alpha < |z| < \beta$ contains no zeros of f.

Let the integer $m \geqq 0$ be the number of zeros, counting multiplicities, which lie in the closed circular interior, $|z| \leqq \alpha$, of the nonconstant function $f(z)$. If $m > 0$, we denote these m zeros, counting multiplicities, by z_h ($h = 1, \cdots, m$). The function

$$A(z) = \prod_1^m {}_h\left(1 - \frac{z_h}{z}\right) \tag{2}$$

(where we understand any such product to have the value 1 if $m = 0$) is then analytic and has no zeros in the open simply-connected circular exterior $|z| > \alpha$ including $z = \infty$. A single-valued function $a(z)$, analytic in this open circular exterior, then exists such that

$$A(z) = a(z)^{2/t}. \tag{3}$$

Since $f(z)$ has no zeros in the open annulus $\alpha < |z| < \beta$, the only zeros of $f(z)$ in the open circular interior $|z| < \beta$ are the zeros z_h ($h = 1, \cdots, m$). Consequently the function

$$B(z) = f(z) \Big/ \prod_1^m {}_h(z - z_h) \tag{4}$$

is analytic and has no zeros in the open simply-connected circular interior $|z|<\beta$. A single-valued function $b(z)$, analytic in this open circular interior, then exists such that

$$B(z) = b(z)^{2/t}. \tag{5}$$

From (2), (3), (4), and (5) we see that

$$f(z) = z^m A(z)B(z) = z^m a(z)^{2/t} b(z)^{2/t} = z^m c(z)^{2/t} \tag{6}$$

where the function $c(z)=a(z)b(z)$ is single-valued and analytic in the open annulus $\alpha<|z|<\beta$ and hence admits a Laurent expansion

$$c(z) = \sum_{-\infty}^{\infty}{}_p c_p z^p \tag{7}$$

in this annulus. In the associated interval $\alpha<r<\beta$ we define

$$s(r) = \frac{1}{2\pi}\int_0^{2\pi} |c(re^{i\theta})|^2 d\theta. \tag{8}$$

This integral (8) may be evaluated in terms of the coefficients of the series (7) according to the well known formula

$$s(r) = \sum_{-\infty}^{\infty}{}_p |c_p|^2 r^{2p}. \tag{9}$$

From (1), (6), and (8) we see that

$$M = r^m s^{1/t}.$$

Differentiating this expression twice with respect to r we obtain the formula

$$\begin{aligned} \mu &= t^2 r^2 s^2 M''/M \\ &= t^2 m(m-1)s^2 + 2tmsrs' + (1-t)r^2 s'^2 + tsr^2 s'', \end{aligned} \tag{10}$$

where primes denote differentiation with respect to r.

Now consider the function

$$N = r^n s^{1/2} \tag{11}$$

where n is an as yet undetermined integer. Differentiating this expression twice with respect to r we obtain the formula

$$\nu = 4r^2 s^2 N''/N = 4n(n-1)s^2 + 4nsrs' - r^2 s'^2 + 2sr^2 s''. \tag{12}$$

Substitution of the power series (9) into (12) gives

$$\nu = \sum_{-\infty}^{\infty}{}_{pq} \nu_{pq} |c_p c_q|^2 r^{2p+2q},$$

where the symmetrized coefficient $\nu_{pq}=\nu_{qp}$ has the value

$$\nu_{pq} = (2n + p + q - 1)^2 + 3(p - q)^2 - 1.$$

By examining the two cases $p=q$ and $p\neq q$ it is easy to see, since n, p, q are integers, that $\nu_{pq}\geqq 0$. Consequently $\nu\geqq 0$ and $N''\geqq 0$.

If $t=2$ and $n=m$, then $N=M$, whence $M''\geqq 0$. Therefore $M_t(r;f)$ is a convex function of r in $\alpha<r<\beta$ if $t=2$.

We now show that $M_t(r;f)$ is a convex function of r in $\alpha<r<\beta$ if $0<t<2$ and if the open interval

$$I_t(m): \quad tm - t < x < tm$$

contains no even integer. Evidently the interval

$$tm - t < x \leqq tm - t + 2$$

contains exactly one even integer $2n$. Since $2n$ does not lie in the interval $I_t(m)$ we conclude that

$$tm \leqq 2n \leqq tm - t + 2,$$

whence

$$tm - 2n \leqq 0 \leqq tm - 2n + 2 - t. \tag{13}$$

Now let this integer n be the n of (11). Consider the following expression quadratic in the variables s and

rs': $\phi = 2\mu - t\nu = 2t(tm(m-1) - 2n(n-1))s^2 + 4t(m-n)srs' + (2-t)r^2s'^2.$

The discriminant of this quadratic form is

$$\Delta = 8t(tm - 2n)(tm - 2n + 2 - t).$$

Since $0<t<2$ the coefficient $2-t$ of $r^2s'^2$ in ϕ is positive, and from (13) we see that $\Delta\leqq 0$. Therefore $\phi\geqq 0$ and, as we have already shown, $\nu\geqq 0$, so we conclude that

$$2\mu = \phi + t\nu \geqq 0,$$

whence $M''\geqq 0$ in the interval $\alpha<r<\beta$.

The open interval $I_t(m)$ contains no even integer $2h$, if t is of the form $2/k$ $(k=2, 3, \cdots)$; else we would have the inequality

$$m - 1 < hk < m,$$

which is impossible since h, k, m are integers. Moreover, it is evident that the open interval $I_t(m)$ contains no even integer if $0<t<2$ and $m=0$ or if $0<t<2/m$ and $m\geqq 1$. Therefore $M_t(r;f)$ is a convex function of r in $\alpha<r<\beta$ if $t=2/k$ $(k=1, 2, 3, \cdots)$ or if $0<t\leqq 2/k$ $(k=1, 2, 3, \cdots)$ and $m\leqq k$.

It is easy to see how the theorem follows from this result. As we have already mentioned, the mean modulus M possesses a continuous derivative M' with respect to r in the interval $0\leqq r<1$ provided $0<t<\infty$. We have shown that under certain conditions M is convex in any open subinterval of $0\leqq r<1$ which contains no moduli of zeros of f; therefore under these conditions M' is nondecreasing in any subinterval closed in $0\leqq r<1$ which contains no moduli of zeros of f in its interior. Since the zeros of the nonconstant analytic function f are isolated, the collection of all such closed subintervals covers the interval $0\leqq r<1$. Thus M' is a nondecreasing function of r and hence M is a convex function of r in this interval.

The theorem is proved except for inclusion of the value $t=0$. However, since $T(f)$ is closed and contains the values $t=2/k$ $(k=1, 2, 3, \cdots)$ it also contains the limit value $t=0$. That $T(f)$ contains the value $t=0$ may also be seen directly from the Jensen formula for the geometric mean.

The following corollary concerns the length $l(r; f)$ of the map under f of the circle $C(r)$.

COROLLARY. *Both the length $l(r; f)$ of the map under f of the circle $C(r)$ and the circular expansion ratio $l(r; f)/2\pi r$ are nondecreasing convex functions of r in the interval $0\leqq r<1$.*

PROOF. It suffices merely to exhibit the following formula for the length of the map of $C(r)$:

$$l(r; f) = \int_0^{2\pi} |f'(re^{i\theta})| \, r d\theta = 2\pi r\cdot M_1(r; f'),$$

where f' is the derivative of f and hence is analytic in the open unit circle.

THEOREM. *If f vanishes at the origin, then $T(f)$ contains the entire parameter interval $0\leqq t\leqq\infty$.*

PROOF. The theorem is evidently true if f is of the form cz where c is a constant. If f is not of this simple form, then, since f vanishes at $z=0$, there exists a nonconstant function g, analytic in the open unit circle, such that

$$f(z) = zg(z).$$

For a fixed parameter t in the interval $0<t<\infty$ let $F=M_t(r; f)$ and $G=M_t(r; g)$, so that

$$F = rG. \tag{14}$$

We shall denote successive differentiations of a function with respect to r by primes, and successive differentiations of the logarithm of a function with respect to $\log r$ by asterisks. In any interval $\alpha < r < \beta$ containing no moduli of zeros of the nonconstant analytic function g the first and second prime and asterisk derivatives of G exist and are connected by the following relations:

$$rG' = GG^*,$$
$$r^2G'' = G(G^{**} + G^{*2} - G^*).$$

Differentiating (14) twice with respect to r and using these relations we obtain the formula

$$rF'' = r^2G'' + 2rG' = G(G^{**} + G^{*2} + G^*).$$

As we have already mentioned $G^* \geqq 0$ and $G^{**} \geqq 0$, so that $F'' \geqq 0$ in $\alpha < r < \beta$. The extension of convexity of F to $0 \leqq r < 1$ proceeds as before. Therefore $T(f)$ contains the interval $0 < t < \infty$ and being closed also includes the limiting parameters $t = 0$ and $t = \infty$.

According to Schwarz's lemma the maximum modulus on the circle $C(r)$ of an analytic function f, which maps the origin into itself and the open unit circle into itself, is not greater than r. The above theorem shows that *this maximum modulus is also a convex function of* r.

Although the mean modulus $M_t(r; f)$ may not be convex in the entire interval $0 \leqq r < 1$, it may be convex in some subinterval containing $r = 0$. We define $\rho(t; f)$ to be the length of the maximal such subinterval if one exists and to be 0 if no such subinterval exists. Since the limit function of convex functions is convex, it is clear that

$$\rho(t; f) \geqq \limsup_{t' \to t} \rho(t'; f),$$

whence $\rho(t; f)$ is an upper semicontinuous function of t.

We now show that $\rho(t; f) > 0$ if $0 \leqq t < \infty$. We have already seen by example that $\rho(\infty; f)$ may be 0. Since $\rho(t; f) = 1$ if $t = 0$, if f is constant, or if f vanishes at $z = 0$, we may suppose that $0 < t < \infty$ and that f is a nonconstant analytic function which does not vanish at $z = 0$. A neighborhood $|z| < \beta$ of $z = 0$ then exists in which f does not vanish. Thus, in the notation of our first theorem we have $m = 0$ and $A(z) = 1$. The expansion (7) is then a Taylor expansion with $c_0 \neq 0$. Moreover, since f is nonconstant, not all the coefficients $c_1, c_2, c_3, \cdots$ vanish. Let $c_q \neq 0$ be the first such nonvanishing coefficient. Substitution of the power series (9) into (10) gives

$$\mu = 2tq(2q - 1) \mid c_0 c_q \mid^2 r^{2q} + O(r^{2q+2}).$$

Since q is a positive integer we infer that $\mu > 0$ and hence $M'' > 0$ in some neighborhood of $r = 0$. Consequently $\rho(t; f) > 0$.

We conclude with the following question: Is $\rho(t; f)$ a continuous, non-increasing function of t?

References

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