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THE UNIVERSITY OF ROCHESTER

ON THE REPRESENTATION OF A FUNCTION AS A HELLINGER INTEGRAL

RICHARD H. STARK

We derive in this note a necessary and sufficient condition that a nondecreasing, continuous function h of a single variable x be representable as a Hellinger integral of the form $\int_0^x (df)^2/dg$. This condition was first proved by Hellinger in his dissertation [1].[1] Other proofs have been given by Hahn [2] and Hobson [3], who transform to Lebesgue integrals and make use of Lebesgue theory. Hellinger's proof and the less complicated proof given here have a certain simplicity in that they avoid reliance on these notions and even remain entirely within the range of monotone functions.

We consider nondecreasing functions of a real variable x on the interval $0 \leqq x \leqq 1$ (henceforth denoted as $[0, 1]$). For such a function $f(x)$ and a closed interval Δ with end points x_1 and x_2 ($x_1 \leqq x_2$), we define a new function $f_\Delta(x)$ to denote the length of the interval on the f-axis determined by the interval on the x-axis common to Δ and $(0, x)$. More precisely, denoting

$$f(x \pm 0) = \lim_{h\to 0} f(x \pm |h|) \qquad \text{if } 0 < x < 1,$$

$$f(0 - 0) = f(0); \qquad f(1 + 0) = f(1),$$

we define

$$f_\Delta(x) = \begin{cases} 0 & \text{if } 0 \leqq x < x_1, \\ f(x + 0) - f(x_1 - 0) & \text{if } x_1 \leqq x \leqq x_2, \\ f(x_2 + 0) & \text{if } x_2 < x \leqq 1. \end{cases} \tag{1}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

The interval $[f(x_1-0), f(x_2+0)]$ on the f-axis will be denoted as $f(\Delta)$. We retain the notation Δf for the proper difference $f(x_2)-f(x_1)$.

Given sets $\{E_i\} \subset [0, 1]$ for which the symbols $f_{E_i}(x)$ and $f(E_i)$ have been defined, we define

$$
(2) \qquad \left.\begin{aligned} f_{\Sigma E_i}(x) &= \sum f_{E_i}(x) \\ f(\sum E_i) &= \sum f(E_i) \end{aligned}\right\} \quad \text{if } E_i \cdot E_j = 0 \text{ when } i \neq j.
$$

$$
\left.\begin{aligned} f_{E_2-E_1}(x) &= f_{E_2}(x) - f_{E_1}(x) \\ f(E_2 - E_1) &= f(E_2) - f(E_1) \end{aligned}\right\} \quad \text{if } E_1 \subset E_2.
$$

The sets which we shall consider will be constructed from sets of intervals in a manner such that (1) and (2) will assign to each such set E a function $f_E(x)$ called the measure function of E on the f-axis and a set $f(E)$ of points on the f-axis. Our $f_E(1)$ is just the measure with respect to $f(x)$ of the set E as defined in [**4**; p. 277]. It is immediate that if $f_{E_D}(x)$ denotes the measure on the $f_E(x)$ axis of the set D on the x-axis, then

$$
(3) \qquad f_{E_D}(x) = f_{E \cdot D}(x).
$$

We now consider functions $f(x)$, $g(x)$, and $h(x)$ which are nondecreasing and continuous in $[0, 1]$ and satisfy the inequality

$$
(4) \qquad (\Delta f)^2 \leqq \Delta g \Delta h \qquad \text{on every subinterval } \Delta \text{ of } [0, 1].
$$

Let M be an arbitrary set on the x-axis. For arbitrary $\epsilon > 0$, the set $g(M)$ on the g-axis may be enclosed in a set I_ϵ of countably many mutually disjoint intervals d_i such that the measures on the g-axis of the sets $g(M)$ and I_ϵ differ by less than ϵ; that is, the inverse function $x(g)$ defines a corresponding set $x(I_\epsilon)$ of the x-axis which we denote for brevity as X_ϵ such that

$$
(5) \qquad 0 \leqq g_{X_\epsilon}(1) - g_M(1) \leqq \epsilon.
$$

Now for Δ an arbitrary interval of the x-axis and Δ_i the intersection of Δ with $x(d_i)$, we have:

$$
(6) \qquad \Delta f_M \leqq \Delta f_{X_\epsilon} = \sum_{i=1}^{\infty} \Delta_i f.
$$

But $(\Delta_i f)^2 \leqq \Delta_i g \Delta_i h$ on every Δ_i so that from the Cauchy-Schwarz inequality:

$$
(7) \qquad \left(\sum_{i=1}^{\infty} \Delta_i f \right)^2 \leqq \sum_{i=1}^{\infty} \Delta_i g \sum_{i=1}^{\infty} \Delta_i h \leqq \Delta g_{X_\epsilon} \Delta h.
$$

With use of (5) and (6), this gives

$$(\Delta f_M)^2 \leqq \Delta g_M \Delta h. \tag{8}$$

Replacing (4) by (8) and carrying out an analogous argument for a set N and the h-axis, we have

$$(\Delta f_{M\cdot N})^2 \leqq \Delta g_M \Delta h_N \qquad \text{if (4) holds.} \tag{9}$$

THEOREM. *To a given pair of nondecreasing, continuous functions* $g(x)$ *and* $h(x)$ *for which* $g(0)=0=h(0)$ *there corresponds an* $f(x)$ *such that*

$$h(x) = \int_0^x \frac{(df)^2}{dg}$$

if and only if for every set E *of the x-axis such that* $g_E(1)=g(1)$, *we have* $h_E(1)=h(1)$.

For the proof, we use the following elementary properties of Hellinger integrals [1] which hold for arbitrary functions $g(x)$ and $h(x)$ that are nondecreasing and continuous in [0, 1]:

(10)
a. Existence of $t(x)=\int_0^x(du)^2/dg$ implies $t(x)=\int_0^x(d\int_0^x(dgdt)^{1/2})^2/dg$.
b. $f(x)=\int_0^x(dgdh)^{1/2}$ exists and $(\Delta f)^2 \leqq \Delta g\Delta h$.
c. The inequality $(\Delta f)^2 \leqq \Delta g \Delta h$ implies that $s(x)=\int_0^x(df)^2/dg$ exists and $(\Delta f)^2/\Delta g \leqq \Delta s \leqq \Delta h$ if $\Delta g \neq 0$.

It follows from (10a) that the desired representation of $h(x)$ exists only if it is given by $\int_0^x(df)^2/dg$ where $f(x)=\int_0^x(dgdh)^{1/2}$. By (10b), $(\Delta f)^2 \leqq \Delta g\Delta h$. Let E be any set such that $g_E(1)=g(1)$. Then from (8), we have on replacing M by the complement $\overline{E}$ of E with respect to [0, 1] and taking for Δ the interval [0, 1] that

$$(f_{\overline{E}}(1))^2 \leqq g_{\overline{E}}(1)h(1) = 0 \tag{11}$$

and consequently $f_E(1)=f(1)$. It follows from application of (11) to (9) that

$$(\Delta f)^2 \leqq \Delta g \Delta h_E. \tag{12}$$

Property (10c) with (12) gives that $\Delta s \leqq \Delta h_E$ so that the function $a(x)=h_E(x)-s(x)$ is nondecreasing and if $\Delta g \neq 0$

$$\begin{aligned} 0 \leqq \Delta a = \Delta h_E - \Delta s &\leqq \Delta h - (\Delta f)^2/\Delta g \\ &= ((\Delta g\Delta h)^{1/2} - \Delta f)((\Delta h/\Delta g)^{1/2} + \Delta f/\Delta g). \end{aligned}$$

Consequently

$$(13)\quad 0 \leqq \Delta a = \Delta h_E - \Delta s \leqq 2(\Delta h/\Delta g)^{1/2}((\Delta g \Delta h)^{1/2} - \Delta f),$$
$$\text{if } g_E(1) = g(1) \text{ and } \Delta g \neq 0.$$

We next choose a sequence of divisions D_n of the interval $(0, 1)$ into finitely many nonoverlapping intervals (that is, no two intervals of D_n have more than an end point in common) such that each D_n is formed from D_{n-1} by addition of finitely many division points and

$$(14)\qquad \sum_{D_n} ((\Delta g \Delta h)^{1/2} - \Delta f) \leqq 4^{-n}.$$

That such a choice is possible is implied by (10a).

For each division D_n, we distinguish two types of intervals:

(15) a. The set G_n of intervals such that $\Delta g \geqq 4^{-n} \Delta h$;
b. The set S_n of intervals such that $\Delta g < 4^{-n} \Delta h$.

Then from (13), (14), (15), we have

$$(16)\qquad a_{G_n}(1) = \sum_{G_n} \Delta a \leqq 2^{-n+1},$$

$$(17)\qquad g_{S_n}(1) = \sum_{S_n} \Delta g \leqq 4^{-n} h(1).$$

We define $R_n = \prod_{m=n}^{\infty} G_m$ and $R = \lim_{n\to\infty} R_n$. It is immediate that

$$(18)\qquad a_R(1) = \lim_{n\to\infty} a_{R_n}(1) \leqq \lim_{n\to\infty} a_{G_n}(1) = 0 \qquad \text{(see (16))},$$

and since $\overline{R} \subset \sum_{m=n}^{\infty} S_m$,

$$g_{\overline{R}}(1) \leqq \lim_{n\to\infty} \sum_{m=n}^{\infty} g_{S_m}(1) = 0 \qquad \text{(see (17))}.$$

Thus R is a set of the type E assumed in (13). Consequently, $0 \leqq \Delta a \leqq \Delta h_R$, and it follows from (9) and (3) that $a_{\overline{R}}(1) \leqq h_{R \cdot \overline{R}}(1) = 0$. Hence

$$(19)\qquad a_{\overline{R}}(1) = 0.$$

The function $a(x)$ is nondecreasing with $a(0) = 0$ and by (18) and (19) has $a(1) = 0$. Thus $a(x) \equiv 0$, that is,

$$(20)\qquad s(x) = h_R(x).$$

Therefore,

$$s(x) = \int_0^x \frac{\left(d \int_0^x (dg dh)^{1/2}\right)^2}{dg}$$

is the measure function on the h-axis of a set R of the x-axis. This integral is actually equal to $h(x)$ if and only if $h_R(1) = h(1)$. The condition that $g_E(1) = g(1)$ implies $h_E(1) = h(1)$ is surely necessary to provide that $h(x) = s(x)$, for $g_E(1) = g(1)$ implies $s_E(1) = s(1)$. On the other hand, if $g_E(1) = g(1)$ does imply $h_E(1) = h(1)$, the condition that $h_R(1) = h(1)$ is fulfilled and $h(x) \equiv s(x)$. This completes the proof.

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Northwestern University