ON THE SUMMABILITY OF CERTAIN ORTHOGONAL DEVELOPMENTS OF NONLINEAR FUNCTIONALS

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1. **Introduction.** Using Wiener's[1] method of integration in function space, Cameron and Martin[2] have constructed a set of closed orthonormal functionals in the space of nonlinear functionals which are Wiener measurable and whose squares are Wiener summable. It is the purpose of this paper to investigate the infinite-dimensional Abel summability of the orthogonal development of a nonlinear functional in terms of these orthogonal functionals.

The space of functions over which Wiener's integral operates is the space C of functions $x(t)$ defined and continuous on $0 \leq t \leq 1$ and vanishing at $t = 0$. The whole space has measure unity, and measure in this space is built up in exactly the same way as ordinary Lebesgue measure, except that the definition of an interval and its measure are different.

Wiener defines an interval Q, or as he calls it, a "quasi-interval" in his space by the inequalities

$$Q: \qquad \alpha_j < x(t_j) < \beta_j, \qquad\qquad j = 1, \cdots, n,$$

where $t_1, \cdots, t_n$ is any finite set of numbers such that $0 < t_1 < t_2 < \cdots < t_n \leq 1$. He defines the measure of Q as

$$m(Q) = \left[\pi^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1})\right]^{-1/2} \int_{\alpha_n}^{\beta_n} \cdots \int_{\alpha_1}^{\beta_1}$$
$$\cdot \exp\left[-\frac{\xi_1^2}{t_1} - \frac{(\xi_2 - \xi_1)^2}{t_2 - t_1} - \cdots - \frac{(\xi_n - \xi_{n-1})^2}{t_n - t_{n-1}} \right] d\xi_1 \cdots d\xi_n,$$

and shows that this definition is self-consistent and leads to a satisfactory measure for general sets, in accordance with the usual definition of Lebesgue measure. After measure is defined, the usual Lebesgue procedure gives a satisfactory definition of integral with all the usual properties except invariance under translations and simple behavior under magnifications. Measurability and summabil-

ity of functionals are defined as usual. It is convenient to denote the Wiener integral of a summable functional $F(x)$ over a measurable set S by

$$\int_S^w F(x)d_w x.$$

In particular, it has been shown that if

$$F(x) = f\left\{\int_0^1 \alpha_1(t)dx(t), \cdots, \int_0^1 \alpha_n(t)dx(t)\right\},$$

where $\{\alpha_j(t)\}$, $j=1, 2, \cdots, n$, are orthonormal and of bounded variation, then provided the second member exists,

(1.1)
$$\int_C^w F(x)d_w x$$
$$= \pi^{-n/2}\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(u_1, \cdots, u_n) \exp\left(\sum_{k=1}^n u_k^2\right)du_1\cdots du_n.$$

Convenient Fubini theorems enable us to interchange order of two Wiener integrals or a Wiener integral and an n-dimensional Lebesgue integral in the ordinary way. Thus the integral has the usual satisfactory manipulative properties. We denote by $L_2(C)$ the space of measurable complex functionals $F(x)$ such that

$$\int_C^w |F(x)|^2 d_w x < \infty.$$

An orthonormal set of functionals $\{F_n\}$ is of course defined to be a set such that

$$\int_C^w F_j(x)\overline{F_k(x)}d_w x = \delta_{jk}, \qquad j, k = 1, 2, \cdots,$$

where δ_{jk} is the Kronecker delta, and such a set is closed if for each functional $G \in L_2(C)$ and a given $\epsilon > 0$, there exists a set of constants $C_1, C_2, \cdots, C_n$ such that

$$\int_C^w \left|G(x) - \sum_{j=1}^n C_j F_j(x)\right|^2 d_w x < \epsilon.$$

A C.O.N. (closed orthonormal) set of functionals can be constructed as follows. Let $\alpha_1(t)$, $\alpha_2(t)$, $\cdots$ be a C.O.N. set of real functions on $0 \le t \le 1$ and each $\alpha_j(t)$ of bounded variation, and let $H_n(u)$ be the

partially normalized Hermite polynomials

$$H_n(u) = (-1)^n 2^{-n/2}(n!)^{-1/2}e^{u^2}\frac{d^n}{du^n}(e^{-u^2}), \qquad n = 0, 1, 2, \cdots .$$

Let

$$\Phi_{m,p}(x) = H_m\left[\int_0^1 \alpha_p(t)dx(t)\right], \qquad m = 0, 1, \cdots ; p = 1, 2, \cdots ,$$

and

$$\Psi_{m_1,\cdots,m_p}(x) \equiv \Psi_{m_1,\cdots,m_p,0,\cdots,0}(x) = \Phi_{m_1,1}(x) \cdots \Phi_{m_p,p}(x).$$

This is called the Fourier-Hermite set of functionals. As the above notation indicates, any number of zero subscripts may be added at the end of the already given list of subscripts without changing Ψ, because $H_0(u) \equiv \Phi_{0,p}(x) \equiv 1$. Thus the entire set of the $\Psi_{m_1,\cdots,m_p}$ without repetition consists of

$\Psi_0;$

$\{\Psi_{m_1}\},$ $m_1 = 1, 2, \cdots ;$

$\{\Psi_{m_1,m_2}\},$ $m_1 = 0, 1, 2, \cdots , m_2 = 1, 2, \cdots ;$

$\cdots \cdots \cdots ;$

$\{\Psi_{m_1,\cdots,m_p}\}, m_i = 0, 1, 2, \cdots$ for $i = 1, 2, \cdots , p - 1, m_p = 1, 2, \cdots ;$

$\cdots \cdots \cdots \cdots .$

Despite our complicated set of subscripts, this is still only a countable set of functionals. It has been shown[2] that this is a C.O.N. set over $L_2(C)$, Thus, if $F(x) \in L_2(C)$, its Fourier-Hermite coefficients are defined to be

$$A_{m_1,\cdots,m_N} = \int_c^w F(x)\Psi_{m_1,\cdots,m_N}(x)d_wx,$$

and we have

$$\lim_{N\to\infty}\int_c^w\left|F(x) - \sum_{m_1,\cdots,m_N=0}^N A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(x)\right|^2 d_wx = 0.$$

Thus the Fourier-Hermite development of F converges in the sense of a limit in the mean for $L_2(C)$ to the original functional. However, this does not tell us that it converges in any sense at all for any given fixed function x, just as convergence in the mean for a Fourier series

does not provide any information at all about the behavior of the series at a specific point. The convergence of the Fourier series at a fixed point requires more specific information about the generating function at that point, and it is reasonable to expect that much more would be required in the case of the functional. It is not at all clear what the analogue of the notion of bounded variation should be in order for us to carry through a pointwise convergence theorem in $L_2(C)$. However, if we are content to establish summability of the Fourier series of a function at the specified point, then we do not need bounded variation at the point, but only continuity; and continuity does have a satisfactory analogue for functionals. Since our orthogonal functionals are given in terms of Hermite functions, our summability will depend on a formula for sums of products of Hermite functions. Such a formula has been given by Wiener,[3] for $|t| < 1$, $-\infty < x < \infty$, $-\infty < y < \infty$, namely,

$$(1.2) \quad \sum_{n=0}^{\infty} t^n \psi_n(x) \psi_n(y) \equiv K(x, y, t)$$

$$= [\pi(1 - t^2)]^{-1/2} \exp\left[\frac{4xyt - (1 + t^2)(x^2 + y^2)}{2(1 - t^2)}\right],$$

where

$$(1.3) \qquad \psi_n(x) = \pi^{-1/4} H_n(x) e^{-x^2/2}, \qquad n = 0, 1, \cdots .$$

Wiener showed that this series converges absolutely when $|t| < 1$ and that for fixed t such that $|t| < 1$, the series converges in the mean in x and y to its value. It also converges in the mean as a function of x for almost all fixed y, and vice versa.

The fact that (1.2) involves powers of t led us to investigate infinite-dimensional Abel summability for the Fourier-Hermite series. We now state our main theorem.

THEOREM. *Let C be the space of all continuous functions $x(t)$ on $0 \leq t \leq 1$ such that $x(0) = 0$. Let $F(x) = F(x(\cdot))$ be a bounded Wiener measurable functional (linear or nonlinear) for $x \in C$, and let $F(x)$ be continuous in the Hilbert topology at $x_0 \in C$, that is,*

$$|F(x) - F(x_0)| < \epsilon \quad when \quad \int_0^1 [x(t) - x_0(t)]^2 dt < \delta^2(\epsilon).$$

Let

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[3] N. Wiener, *The Fourier integral*, pp. 55–67.

$$\Psi_{m_1,\cdots,m_N}(x) = H_{m_1}\left(\int_0^1 2^{1/2}\cos\frac{\pi t}{2}\,dx(t)\right)$$

$$\cdot H_{m_2}\left(\int_0^1 2^{1/2}\cos\frac{3\pi t}{2}\,dx(t)\right)$$

$$\cdots H_{m_N}\left(\int_0^1 2^{1/2}\cos\frac{(2N-1)\pi t}{2}\,dx(t)\right),$$

and let

$$A_{m_1,\cdots,m_N} = \int_C^w F(x)\Psi_{m_1,\cdots,m_N}(x)d_wx;$$

then

$$F(x_0) = \lim_{N\to\infty,\lambda\to1-0}\sum_{m_1,\cdots,m_N=0}^\infty A_{m_1,\cdots,m_N}\lambda^{m_1+\cdots+m_N}\Psi_{m_1,\cdots,m_N}(x_0).$$

(Remark. The reader will note that we have specialized the functional Ψ by taking the C.O.N. set $\{\alpha_j(t)\}$ to be

(1.4) $$\alpha_j(t) = 2^{1/2}\cos\frac{(2j-1)\pi t}{2}, \qquad j = 1, 2, \cdots.$$

It was found necessary to take advantage of the special properties of the $\cos(j-1/2)\pi t$. In particular, these functions satisfy the boundary conditions $\alpha_j(1) = \alpha_j'(0) = 0$. It is hoped that later investigations will make it possible to use less specialized $\alpha_j(t)$.)

Using the fact that the Abel mean is regular for one-dimensional series we shall prove also the following corollary on pointwise convergence:

COROLLARY. *If we assume, in addition to the hypotheses of the theorem, that for each positive integer N*

(1.5) $$\sum_{m_1,\cdots,m_N=0}^\infty A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(x_0)$$

converges absolutely, then

(1.6) $$F(x_0) = \lim_{N\to\infty}\sum_{m_1,\cdots,m_N=0}^\infty A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(x_0).$$

2. An integral expression for the difference $D(x_0)$ between a functional and its Fourier-Hermite partial sum. In order to prove our main theorem stated above we proceed to sum the Fourier-Hermite

series with appropriate powers of λ as the weight factors. Putting $x = y$ and $t = \lambda^2$ in (1.2) we get

$$K(x,\ x,\ t) = \sum_{n=0}^{\infty} \lambda^{2n} \psi_n^2(x) = \sum_{n=0}^{\infty} \lambda^{2n} \pi^{-1/2} e^{-x^2} H_n^2(x),$$

and hence the series

$$\sum_{n=0}^{\infty} \lambda^{2n} H_n^2(x) = \pi^{1/2} e^{x^2} K(x,\ x,\ t)$$

converges absolutely when $|\lambda| < 1$. If we replace x by $\int_0^1 \alpha_j(t) dx_0(t)$ and understand by $\alpha_j(t)$ the specialized functions of (1.4), then when $x \in C$,

(2.1)

$$\prod_{j=1}^{N} \left[\sum_{m_j=0}^{\infty} \lambda^{2m_j} H_{m_j}^2 \left(\int_0^1 \alpha_j(t) dx_0(t) \right) \right]$$

$$= \sum_{m_1, \cdots, m_N = 0}^{\infty} \lambda^{2(m_1 + \cdots + m_N)} H_{m_1}^2 \left(\int_0^1 \alpha_1(t) dx_0(t) \right)$$

$$\cdots H_{m_N}^2 \left(\int_0^1 \alpha_N(t) dx_0(t) \right)$$

$$= \sum_{m_1, \cdots, m_N = 0}^{\infty} \lambda^{2(m_1 + \cdots + m_N)} \Psi_{m_1, \cdots, m_N}^2(x_0)$$

converges absolutely for $|\lambda| < 1$. Since the functionals $\{\Psi_{m_1, \cdots, m_N}(x)\}$ form a C.O.N. set in the space $L_2(C)$, we may use the Riesz-Fischer theorem for that space. Thus

(2.2) $\underset{M \to \infty}{L.I.M.} \sum_{m_1, \cdots, m_N = 0}^{M} \lambda^{m_1 + \cdots + m_N} \Psi_{m_1, \cdots, m_N}(x_0) \Psi_{m_1, \cdots, m_N}(x)$

exists, since (2.1) shows that the sum of the squares of the coefficients of $\Psi_{m_1, \cdots, m_N}(x)$ in (2.2) converges. Since $F(x)$ is bounded and Wiener measurable, $F(x) \subset L_2(C)$, and this with the existence of (2.2) justifies the interchange of order of integration and summation in:

$$\int_c^w F(x) \sum_{m_1, \cdots, m_N = 0}^{\infty} \lambda^{m_1 + \cdots + m_N} \Psi_{m_1, \cdots, m_N}(x) \Psi_{m_1, \cdots, m_N}(x_0) d_w x$$

(2.3) $= \sum_{m_1, \cdots, m_N = 0}^{\infty} \lambda^{m_1 + \cdots + m_N} \int_c^w F(x) \Psi_{m_1, \cdots, m_N}(x) d_w x \Psi_{m_1, \cdots, m_N}(x_0)$

$$= \sum_{m_1, \cdots, m_N = 0}^{\infty} \lambda^{m_1 + \cdots + m_N} A_{m_1, \cdots, m_N} \Psi_{m_1, \cdots, m_N}(x_0).$$

Thus the convergence of the last member of (2.3) for all x_0 and for $|\lambda| < 1$ is established. In the same way that we proved the absolute convergence of the last member of (2.1) we can show that the first member (as well as the second and third) of the following expression converges absolutely:

$$\sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N}\Psi_{m_1,\ldots,m_N}(x)\Psi_{m_1,\ldots,m_N}(x_0)$$

$$= \sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N}H_{m_1}\left(\int_0^1 \alpha_1(t)dx(t)\right)$$

(2.4) $$\cdots H_{m_N}\left(\int_0^1 \alpha_N(t)dx(t)\right)$$

$$\cdot H_{m_1}\left(\int_0^1 \alpha_1(t)dx_0(t)\right)\cdots H_{m_N}\left(\int_0^1 \alpha_N(t)dx_0(t)\right)$$

$$= \prod_{j=1}^{N}\left[\sum_{m_j=0}^{\infty}\lambda^{m_j}H_{m_j}\left(\int_0^1 \alpha_j(t)dx(t)\right)\right.$$

$$\left.\cdot H_{m_j}\left(\int_0^1 \alpha_j(t)dx_0(t)\right)\right].$$

In view of (1.2) and (1.3) we have

$$K(x,\ y,\ t) = \sum_{n=0}^{\infty} t^n\pi^{-1/2}e^{-(x^2+y^2)/2}H_n(x)H_n(y),$$

so that

$$\sum_{n=0}^{\infty} t^nH_n(x)H_n(y) = \pi^{1/2}e^{(x^2+y^2)/2}K(x,\ y,\ t)$$

(2.5)

$$= (1-t^2)^{-1/2}\exp\left[\frac{2xyt - t^2(x^2+y^2)}{1-t^2}\right].$$

Thus replacing t, x, y by λ, $\int_0^1\alpha_j(t)dx_0(t)$, $\int_0^1\alpha_j(t)dx(t)$ respectively in (2.5) and substituting in (2.4) we get

$$\sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N}\Psi_{m_1,\ldots,m_N}(x)\Psi_{m_1,\ldots,m_N}(x_0)$$

(2.6)

$$= \prod_{i=1}^{N}\left[(1-\lambda^2)^{-1/2}\exp\left\{\frac{2\lambda u_i\xi_i - \lambda^2(u_i^2+\xi_i^2)}{1-\lambda^2}\right\}\right],$$

where

$$(2.7) \qquad u_i = \int_0^1 \alpha_i(t)dx(t), \quad \xi_i = \int_0^1 \alpha_i(t)dx_0(t), \qquad i = 1, 2, \cdots.$$

We now substitute (2.6) and (2.3) and obtain

$$(2.8) \quad \begin{aligned} & \sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N} A_{m_1,\cdots,m_N} \Psi_{m_1,\cdots,m_N}(x_0) \\ & = (1-\lambda^2)^{-N/2} \int_C^w F(x) \exp\left[\sum_{i=1}^N \frac{2\lambda u_i\xi_i - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2}\right] d_w x. \end{aligned}$$

If we specialize $F(x)$ for the moment by making $F(x) \equiv 1$, then since $\Psi_{0,0,\cdots,0} = \Psi_0 = H_0 \equiv 1$, and all the other Ψ's are orthogonal to Ψ_0, we have $A_{0,0,\cdots,0} = 1$, and all the other A's are zero. Hence

$$(2.9) \quad (1-\lambda^2)^{-N/2} \int_C^w 1 \cdot \exp\left[\sum_{i=1}^N \frac{2\lambda u_i\xi_i - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2}\right] d_w x = 1$$

with u_i given by (2.7). Then we may multiply through by $F(x_0)$ in (2.9), subtract from (2.8), and obtain

$$(2.10) \quad \begin{aligned} D(x_0) & \equiv \sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N} A_{m_1,\cdots,m_N} \Psi_{m_1,\cdots,m_N}(x_0) - F(x_0) \\ & = c_\lambda \int_C^w [F(x) - F(x_0)] \exp\left[\sum_{i=1}^N \frac{2\lambda u_i\xi_i - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2}\right] d_w x, \end{aligned}$$

where

$$(2.11) \qquad\qquad c_\lambda = (1-\lambda^2)^{-N/2}.$$

We have now obtained an integral expression for the difference $D(x_0)$ between a functional and its Fourier-Hermite partial sum, and we shall establish in the following sections that $D(x_0) \to 0$ as $\lambda \to 1-0$ and $N \to \infty$.

3. **Domination of $D(x_0)$ by a sum of integrals I.** To prove that $D(x_0) \to 0$ as $\lambda \to 1-0$ and $N \to \infty$, we first choose an arbitrary $\epsilon > 0$, then break up our space C and express it as a sum of sets S_j characterized by certain inequalities, and finally dominate $D(x_0)$ by a sum of integrals I_j taken over the sets S_j.

Choose $\epsilon > 0$. Then there exists a $\delta > 0$ such that if

$$(3.1) \qquad \int_0^1 [x(t) - x_0(t)]^2 dt \leq \delta^2, \quad |F(x) - F(x_0)| < \epsilon/3.$$

We let S_0 denote the set of functions $x(t)$ satisfying this first inequality:

$$S_0 = \operatorname*{E}_{x(t)} \left\{ \int_0^1 [x(t) - x_0(t)]^2 dt \leqq \delta^2 \right\}.$$

The complement S_0^{-1} of S_0 will then consist of all $x(t)$ satisfying the reverse inequality:

$$(3.2) \qquad S_0^{-1} = \operatorname*{E}_{x(t)} \left\{ \int_0^1 [x(t) - x_0(t)]^2 dt > \delta^2 \right\}.$$

Let

$$\mu = \frac{\delta}{2} \left(1 + \int_0^1 x_0^2(t) dt \right)^{-1/2},$$

so that

$$(3.3) \qquad \mu^2 \int_0^1 x_0^2(t) dt < \frac{\delta^2}{4}.$$

Choose N_0 so great that

$$(3.4) \qquad \sum_{j=N_0+1}^{\infty} \left(\int_0^1 \beta_j(t) x_0(t) dt \right)^2 < \frac{\delta^2}{16},$$

where

$$(3.5) \qquad \beta_j(t) = 2^{1/2} \sin (j - 1/2)\pi t, \qquad j = 1, 2, \cdots .$$

This can be done since this sum is just the latter part of the (necessarily convergent) Parseval sum for $\int_0^1 x_0^2(t) dt$. Now let $x_1(t)$ be the finite sum

$$x_1(t) = \sum_{j=1}^{N_0} \beta_j(t) \int_0^1 \beta_j(s) x_0(s) ds,$$

and $x_2(t) = x_0(t) - x_1(t)$. Thus

$$(3.6) \qquad \int_0^1 \beta_j(t) x_2(t) dt = \begin{cases} 0 & \text{if } j \leqq N_0 \\ \int_0^1 \beta_j(t) x_0(t) dt & \text{if } j > N_0, \end{cases}$$

$$(3.7) \qquad \int_0^1 x_2^2(t) dt = \sum_{j=N+1}^{\infty} \left[\int_0^1 \beta_j(t) x_0(t) dt \right]^2 < \frac{\delta^2}{16}.$$

We now have from (3.2), (3.3), and (3.7) that when $x(t) \in S_0^{-1}$,

$$\int_0^1 [x(t) - x_0(t)]^2 dt - \mu^2 \int_0^1 x_0^2(t) dt - 4 \int_0^1 x_2^2(t) dt > \frac{\delta^2}{2},$$

or by the Parseval equation

$$\sum_{j=1}^\infty \left[\left\{ \int_0^1 \beta_j(t)[x(t) - x_0(t)] dt \right\}^2 - \mu^2 \left\{ \int_0^1 \beta_j(t) x_0(t) dt \right\}^2 \right.$$
$$\left. - 4 \left\{ \int_0^1 \beta_j(t) x_2(t) dt \right\}^2 \right] > \frac{\delta^2}{2},$$

and by integration by parts (using (1.4) and (3.5)),

$$\sum_{j=1}^\infty \frac{\left\{ \int_0^1 \alpha_j(t) d[x(t) - x_0(t)] \right\}^2 - \mu^2 \left\{ \int_0^1 \alpha_j(t) dx_0(t) \right\}^2 - 4 \left\{ \int_0^1 \alpha_j(t) dx_2(t) \right\}^2}{\pi^2 (j - 1/2)^2} > \frac{\delta^2}{2}.$$

Note that the sum on the left would still converge if the numerator were independent of j and the denominator involved $(j-1/2)$ to the 3/2 power only. Let us therefore investigate the least upper bound of the product of the numerator and $(j-1/2)^{-1/2}$: that is, let

$$\eta = \sup_j \left[\frac{\left\{ \int_0^1 \alpha_j(t) d[x(t) - x_0(t)] \right\}^2 - \mu^2 \left\{ \int_0^1 \alpha_j(t) dx_0(t) \right\} - 4 \left\{ \int_0^1 \alpha_j(t) dx_2(t) \right\}^2}{(j - 1/2)^{1/2}} \right].$$

Then we have (whether η is finite or $+\infty$)

$$\sum_{j=1}^\infty \frac{\eta}{\pi^2 (j - 1/2)^{3/2}} > \frac{\delta^2}{2},$$

and

$$\eta > \frac{\pi^2 \delta^2}{2 \sum_{j=1}^\infty \left(j - \frac{1}{2} \right)^{-3/2}}.$$

But

$$\sum_{j=1}^\infty \left(j - \frac{1}{2} \right)^{-3/2} = \left(\frac{1}{2} \right)^{-3/2} + \sum_{j=2}^\infty \left(j - \frac{1}{2} \right)^{-3/2}$$
$$< 2^{3/2} + \int_1^\infty \left(s - \frac{1}{2} \right)^{-3/2} ds$$
$$= 2^{3/2} + 2^{3/2} < 6,$$

so

$$\eta > \pi^2\delta^2/12 > \delta^2/2.$$

By definition of η, whether η is finite or infinite, there must exist some value of i, say $i=i_0$, for which

$$\frac{\delta^2}{2} < \frac{\{\int_0^1 \alpha_i(t) d[x(t)-x_0(t)]\}^2 - \mu^2\{\int_0^1 \alpha_i(t)dx_0(t)\}^2 - 4\{\int_0^1 \alpha_i(t)dx_2(t)\}^2}{(i-1/2)^{1/2}},$$

so that if

$$(3.8) \quad u_i = \int_0^1 \alpha_i(t)dx(t), \quad \xi_i = \int_0^1 \alpha_i(t)dx_0(t), \quad \xi_i' = \int_0^1 \alpha_i(t)dx_2(t),$$

we have for $x \in S_0^{-1}$ and some corresponding value of i (namely, the i_0 chosen above),

$$(u_i - \xi_i)^2 > \frac{\delta^2}{2}\left(i - \frac{1}{2}\right)^{1/2} + \mu^2\xi_i^2 + 4\xi_i'^2.$$

Since

$$(a + b + c)^2 \leqq 3(a^2 + b^2 + c^2),$$

we have

$$|u_i - \xi_i| > \frac{1}{3^{1/2}}\left[\frac{\delta}{2^{1/2}}\left(i - \frac{1}{2}\right)^{1/4} + \mu|\xi_i| + 2|\xi_i'|\right]$$

$$> \frac{\delta}{3}\left(i - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu|\xi_i| + |\xi_i'|.$$

Thus if we denote by S_j the set

$$(3.9) \quad S_j = \operatorname*{E}_{x(t)}\left\{\left|\int_0^1 \alpha_j(t)dx(t) - \xi_j\right| > \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu|\xi_j| + |\xi_j'|\right\}$$

for $j=1, 2, 3, \cdots$, we have

$$S_0^{-1} \subset \sum_{j=1}^{\infty} S_j,$$

and

$$(3.10) \quad C = S_0 + \sum_{j=1}^{\infty} S_j = \sum_{j=0}^{\infty} S_j.$$

Now if we let

$$
I_j = c_\lambda \int_{S_j}^{w} |F(x) - F(x_0)|
$$

(3.11)

$$
\cdot \exp \left\{ \sum_{i=1}^{N} \frac{2\lambda u_i \xi_i - \lambda^2 (u_i^2 + \xi_i^2)}{1 - \lambda^2} \right\} d_w x,
$$

then we have by (2.10) and (3.10),

(3.12)
$$
|D(x_0)| \leq \sum_{j=0}^{\infty} I_j.
$$

In the following sections we shall obtain suitable estimates for the integrals I_j.

4. One-dimensional integral expressions for the Wiener integrals I_j. It is easy to see from (3.1), (2.9), and (2.11) that for I_0 we shall have

$$
I_0 < \frac{\epsilon}{3} c_\lambda \int_{S_0}^{w} \exp \left\{ \sum_{i=1}^{N} \frac{2u_i \xi_i \lambda - \lambda^2 (u_i^2 + \xi_i^2)}{1 - \lambda^2} \right\} d_w x
$$

(4.1)

$$
\leq \frac{\epsilon}{3} c_\lambda \int_{C}^{w} \exp \left\{ \sum_{i=1}^{N} \frac{2u_i \xi_i \lambda - \lambda^2 (u_i^2 + \xi_i^2)}{1 - \lambda^2} \right\} d_w x = \frac{\epsilon}{3} .
$$

Consider now I_j, for $j > 0$. Since $F(x)$ is bounded by hypothesis, there exists a number B such that $|F(x)| \leq B$. Then from (3.11)

$$
I_j \leq 2c_\lambda B \int_{S_j}^{w} \exp \left\{ \sum_{i=1}^{N} \frac{2\lambda u_i \xi_i - \lambda^2 (u_i^2 + \xi_i^2)}{1 - \lambda^2} \right\} d_w x
$$

$$
= 2c_\lambda B \int_{S_j}^{w} \exp \left\{ \sum_{i=1}^{N} u_i^2 \right\} \exp \left\{ - \sum_{i=1}^{N} \frac{(\lambda \xi_i - u_i)^2}{1 - \lambda^2} \right\} d_w x.
$$

The Wiener integral on the right depends on the u_i as $i = 1, 2, \cdots, N$ and on u_j; that is, it depends on N linear Stieltjes integrals

$$
u_k = \int_0^1 \alpha_k(t) dx(t), \qquad k = 1, 2, \cdots, N,
$$

if $j \leq N$, while if $j > N$, it depends also on the additional

$$
u_j = \int_0^1 \alpha_j(t) dx(t).
$$

These Wiener integrals can be evaluated by (1.1), and present correspondingly an N- or $(N+1)$-fold Lebesgue integral. Let now

$$(4.2) \quad \chi_i(v) = \begin{cases} 1, & \text{if } |v - \xi_i| > \dfrac{\delta}{3}\left(j - \dfrac{1}{2}\right)^{1/4} + \dfrac{1}{2}\mu|\xi_i| + |\xi_j'|, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have by (3.9), (3.8), and (4.2)

$$(4.3) \quad \begin{aligned} I_j &\leq 2Bc_\lambda \pi^{-(N+1)/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_j(v_j) e^{-v_j^2} \\ &\cdot \exp\left\{-\sum_{i=1}^{N} \frac{(\lambda\xi_i - v_i)^2}{1 - \lambda^2}\right\} dv_1 \cdots dv_N dv_j \qquad (j > N), \end{aligned}$$

$$(4.4) \quad \begin{aligned} I_j &\leq 2Bc_\lambda \pi^{-N/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_j(v_j) \\ &\cdot \exp\left\{-\sum_{i=1}^{N} \frac{(\lambda\xi_i - v_i)^2}{1 - \lambda^2}\right\} dv_1 \cdots dv_N \qquad (j \leq N). \end{aligned}$$

In the case of the $(N+1)$-fold integral of (4.3), we have N factors of the form

$$(4.5) \quad \int_{-\infty}^{\infty} \exp\left\{-\frac{(\lambda\xi_i - v)^2}{1 - \lambda^2}\right\} dv$$

and one of the form

$$\int_{-\infty}^{\infty} \chi_j(v) e^{-v^2} dv.$$

For the N-fold integral of (4.4), there are $N-1$ factors of the type (4.5) and the other one (for v_j) is

$$\int_{-\infty}^{\infty} \chi_j(v) \exp\left\{-\frac{(\lambda\xi_j - v)^2}{1 - \lambda^2}\right\} dv.$$

The integrals of type (4.5) are easily evaluated, and we get for each one $(\pi(1-\lambda^2))^{1/2}$. Thus we have (remembering (2.11)),

$$(4.6) \quad I_j \leq \frac{2B}{\pi^{1/2}} \int_{-\infty}^{\infty} \chi_j(v) e^{-v^2} dv \qquad (j > N),$$

$$(4.7) \quad I_j \leq \frac{2B}{(\pi(1 - \lambda^2))^{1/2}} \int_{-\infty}^{\infty} \chi_j(v) \exp\left\{-\frac{(\lambda\xi_j - v)^2}{1 - \lambda^2}\right\} dv \qquad (j \leq N).$$

5. **Estimation of I_j for the case $j > N$.** In this section we shall refine our estimate for I_j for $j > N$. We take now $N > N_0$ (see (3.4)) and note that since $j > N > N_0$, we have, by (3.6) and (3.8), $\xi_j' = \xi_j$, and hence by (4.2) if $\chi_j(v) = 1$,

$$| v - \xi_j | > \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu | \xi_j | + | \xi_j |$$

$$\geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + | \xi_j |.$$

Thus, $\chi_j(v) = 1$ implies that

$$(5.1) \qquad\qquad | v | \geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4},$$

so that we don't decrease the integral in (4.6) when we drop $\chi_j(v)$ and integrate over the v defined by (5.1). Thus

$$(5.2) \quad I_j \leqq \frac{2B}{\pi^{1/2}} \cdot 2 \int_{\delta(j-1/2)^{1/4}/3}^{\infty} e^{-v^2}dv = \frac{4B}{\pi^{1/2}} \operatorname{erfc}\left\{\frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4}\right\},$$

where

$$\operatorname{erfc}(M) = \int_{M}^{\infty} e^{-t^2}dt.$$

We note that this estimate of I_j is *independent* of λ.

6. **Estimation of I_j for the case $j \leqq N$.** In order to reffne our estimate of I_j in this case we make a change of variable in (4.7):

$$s = \frac{v - \lambda\xi_j}{(1 - \lambda^2)^{1/2}},$$

so that the integral of (4.7) becomes

$$(6.1) \quad \int_{-\infty}^{\infty} \chi_j(v) \exp\left\{-\frac{(v - \lambda\xi_j)^2}{1 - \lambda^2}\right\} dv$$

$$= (1 - \lambda^2)^{1/2} \int_{-\infty}^{\infty} \chi_j(s(1 - \lambda^2)^{1/2} + \lambda\xi_j)e^{-s^2}ds.$$

If $1 > \lambda > 1 - \mu/2$, then

$$(1 - \lambda) | \xi_j | \leqq 2^{-1}\mu | \xi_j |,$$

and if $\chi_j(s(1 - \lambda^2)^{1/2} + \lambda\xi_j) = 1$, we have from (4.2)

$$\left| \, s(1 - \lambda^2)^{1/2} + (\lambda - 1)\xi_j \, \right| > \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{\mu}{2}\left| \, \xi_j \, \right| + \left| \, \xi_j' \, \right|$$

$$\geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{\mu}{2}\left| \, \xi_j \, \right|.$$

Thus

(6.2) $$\left| \, s(1 - \lambda^2)^{1/2} \, \right| \geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4},$$

that is,

$$\left| \, s \, \right| \geqq \frac{\delta(j - 1/2)^{1/4}}{3(1 - \lambda^2)^{1/2}},$$

so that we don't decrease the integral in (6.1) when we drop χ_j and integrate over the set of s defined by (6.2). Hence we obtain for $j \leqq N$ and $1 > \lambda > 1 - \mu/2$

(6.3) $$I_j \leqq \frac{4B}{\pi^{1/2}} \int_{\delta(j-1/2)^{1/4}/3(1-\lambda^2)^{1/2}}^{\infty} e^{-s^2} ds = \frac{4B}{\pi^{1/2}} \, \mathrm{erfc} \, \left\{ \frac{\delta(j - 1/2)^{1/4}}{3(1 - \lambda^2)^{1/2}} \right\}.$$

7. The final estimate for $D(x_0)$. We now combine our estimates for I_j, making both assumptions of §§5 and 6; in particular, that $N > N_0$ and $1 > \lambda > 1 - \mu/2$. Then (5.2) holds for $j > N$ and (6.3) for $j \leqq N$. We have then

$$\sum_{j=1}^{\infty} I_j \leqq \frac{4B}{\pi^{1/2}}\left[\sum_{j=1}^{N} \mathrm{erfc} \, \left\{ \frac{\delta(j - 1/2)^{1/4}}{3(1 - \lambda^2)^{1/2}} \right\} \right.$$

$$\left. + \sum_{j=N+1}^{\infty} \mathrm{erfc} \, \left\{ \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} \right\} \right].$$

Since

$$\mathrm{erfc} \, (M) = \int_{M}^{\infty} e^{-t^2} dt < \frac{e^{-M^2}}{2M} \qquad\qquad (M > 0)$$

it follows that

(7.1) $$\sum_{j=1}^{\infty} I_j < \frac{2B}{\pi^{1/2}} \sum_{j=1}^{\infty} \frac{3(1 - \lambda^2)^{1/2}}{\delta(j - 1/2)^{1/4}} \exp \left\{ \frac{- \delta^2(j - 1/2)^{1/2}}{9(1 - \lambda^2)} \right\}$$

$$+ \frac{2B}{\pi^{1/2}} \sum_{j=N+1}^{\infty} \frac{3}{\delta(j - 1/2)^{1/4}} \exp \left\{ \frac{- \delta^2(j - 1/2)^{1/2}}{9} \right\}.$$

But the sum in the second term on the right above converges, so that we may take N_ϵ ($\geq N_0$) large enough to make this term less than $\epsilon/3$ for $N > N_\epsilon$. Moreover, the series of the first term on the right converges for all λ, $0 < \lambda < 1$, and each term of this sum is decreasing in λ as $\lambda \to 1 - 0$, and has the limit zero. Thus the first term on the right of (7.1) has the limit zero. We may therefore choose λ_ϵ in the interval $(1 - \mu/2, 1)$ so that the first term is less than $\epsilon/3$ when $\lambda_\epsilon < \lambda < 1$. Thus when $N > N_\epsilon$ and $\lambda_\epsilon < \lambda < 1$ we have from (3.12) and (4.1),

$$\left| D(x_0) \right| \leq \sum_{j=0}^{\infty} I_j < \epsilon.$$

Referring to the theorem of §1 and the definition of $D(x_0)$ in (2.10), we see that the theorem is established.

8. **Proof of the corollary.** The regularity of the Abel mean for the one-dimensional case, together with the absolute convergence of (1.5) assumed in the hypothesis of the corollary, at once permit us to write

$$\lim_{\lambda \to 1 - 0} \sum_{n=0}^{\infty} \lambda^n \sum_{m_1 + \cdots + m_N = n} A_{m_1, \ldots, m_N} \Psi_{m_1, \ldots, m_N}(x_0)$$

$$= \sum_{n=0}^{\infty} \sum_{m_1 + \cdots + m_N = n} A_{m_1, \ldots, m_N} \Psi_{m_1, \ldots, m_N}(x_0)$$

$$= \sum_{m_1, \ldots, m_N = 0}^{\infty} A_{m_1, \ldots, m_N} \Psi_{m_1, \ldots, m_N}(x_0).$$

Now let $\epsilon > 0$. Then by the theorem there exists an $N_0 > 0$ and a λ_0 in $(0, 1)$ such that when $N > N_0$ and $\lambda_0 < \lambda < 1$,

$$(8.1) \quad \left| F(x_0) - \sum_{m_1, \ldots, m_N = 0}^{\infty} A_{m_1, \ldots, m_N} \lambda^{m_1 + \cdots + m_N} \Psi_{m_1, \ldots, m_N}(x_0) \right| < \epsilon.$$

Taking the limit with respect to λ in (8.1) we get

$$\left| F(x_0) - \sum_{m_1, \ldots, m_N = 0}^{\infty} A_{m_1, \ldots, m_N} \Psi_{m_1, \ldots, m_N}(x_0) \right| \leq \epsilon.$$

But this establishes (1.6), and hence the corollary.

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