NOTE ON HOMOGENEOUS PLANE CONTINUA

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In his Doctoral Dissertation (Texas, 1947), E. E. Moise proved that there exists a compact plane continuum M (not an arc) which is homeomorphic to each of its subcontinua [1].[1] Subsequently, R. H. Bing showed that M is homogeneous [2]. Bing's result flatly contradicts the previously announced result of G. Choquet to the effect that a homogeneous, compact, plane continuum must be a simple closed curve [3]. It is the purpose of this note to show that had Choquet assumed in addition to homogeneity that the continuum was aposyndetic[2] at some point, or that some point of the continuum failed to be a weak cut point[3] of it, then his conclusion would have been valid.

THEOREM 1. *If a compact, plane continuum M is both homogeneous and aposyndetic, then M is a simple closed curve.*

PROOF. If a point of M is of order 2 in M, then M is a simple closed curve [4]. So suppose that *no* point of M is of order 2 in M.

Let G denote the collection of all the complementary domains of M. Because M is homogeneous and contains a non-separating point, no point of M separates M. Since M is aposyndetic, M is semi-locally-connected [5]. Hence each element of G is a simple domain [6, 7]. Let the simple closed curve J denote the boundary of an element D of G.

Case 1. If $M-J$ is connected, then each point of M belongs to some such simple closed curve lying in M. Consequently each point of M belongs to the boundary of an element of G. But G is countable. Hence $M=\sum J_i$ ($i=1, 2, 3, \cdots$), such that for each i, J_i is the boundary of an element of G. Since no point of M is of order 2 in M, each point of J_i is a limit point of $M-J_i$. This contradicts a well known theorem (Baire).

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

[2] A continuum M is said to be *aposyndetic* at a point x of M if for each point y of $M-x$, there exists a subcontinuum of M and an open subset U of M such that $M-y\supset H\supset U\supset x$. If a continuum is aposyndetic at each of its points, then it is said to be aposyndetic.

[3] A point p of a continuum M is a *weak cut point* of M if $M-p$ is not strongly (continuum-wise) connected. For other definitions the reader is referred to Moore's book or Whyburn's book, volumes 13 and 28, respectively, of the American Mathematical Society Colloquium Publications.

Case 2. If $M-J$ is not connected, then $M-J=H+K$ such that $H\cdot\overline{K}=\overline{H}\cdot K=0$. There exists an arc T lying in the complement of $\overline{D}$ which is irreducible from H to K. Except for its end points, T lies in an element U of G. Let C denote the boundary of U and let the point x of C be one of the end points of T. The component AxB of $C-C\cdot J$ which contains x is an arc-segment whose end points lie on J. It follows that $M-(A+B)$ is *not* connected, but $M-B$ *is* connected. Hence A is a local separating point of M. Since M is homogeneous, every point of M is a local separating point of M. But all except countably many of the local separating points of M must be of order 2 in M [8].

THEOREM 2. *If a homogeneous, compact, plane continuum M contains no weak cut point, it is a simple closed curve.*

PROOF. If M contains no weak cut point, it is aposyndetic at some point [9]. Being aposyndetic at one point, M must be aposyndetic at each of its points. Hence Theorem 2 follows from Theorem 1.

BIBLIOGRAPHY

1. E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua.* Trans. Amer. Math. Soc. vol. 63 (1948) pp. 581–594.

2. R. H. Bing, *A homogeneous indecomposable plane continuum,* Bull. Amer. Math. Soc. Abstract 53-5-268.

3. G. Choquet, *Prolongement d'homeomorphies. Ensembles topologiquement nommables. Characterisation topologique individuelle des ensembles fermes totalement discontinuum,* C. R. Acad. Sci. Paris vol. 219 (1944) pp. 542–544, as reported in Mathematical Reviews vol. 7 (1946) p. 335.

4. Karl Menger, *Kurventheorie,* Teubner, 1932, p. 267.

5. F. B. Jones, *Concerning aposyndetic continua and certain boundary problems,* Amer. J. Math. vol. 58 (1941) pp. 545–553, Theorem 4.

6. R. L. Wilder, *Sets which satisfy certain avoidability conditions,* Časopis pro Pěstování Mathematiky a Fysiky vol. 67 (1938) pp. 185–198.

7. G. T. Whyburn, *Semi-locally-connected sets,* Amer. J. Math. vol. 61 (1939) pp. 733–749.

8. G. T. Whyburn, *Analytic topology,* Amer. Math. Soc. Colloquium Publications, vol. 28, Theorem 9.1, p. 61.

9. F. B. Jones, *Concerning non-aposyndetic continua,* Amer. J. Math. vol. 70 (1948) pp. 403–413, Theorem 18.

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