TOPOLOGICAL GROUPS AND GENERALIZED MANIFOLDS

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In a recent paper [4],[1] Montgomery showed that in a locally euclidean 3-dimensional group, any 2-dimensional closed subgroup is also locally euclidean. In this note we prove an analogous result for higher dimensions and more general spaces.

THEOREM. *Let G be a locally compact space which is both a topological group and an n-dimensional orientable generalized manifold. Let H be a closed connected $(n-1)$-dimensional subgroup. Then, if H carries a nonbounding $(n-1)$-cycle, H is also an orientable generalized manifold.*

The terminology used in the statement of this theorem, and in what follows, is that of our two previous papers on generalized manifolds [**1**, **2**], and we assume that the reader is familiar with them.

We make, however, one change. We find it convenient to define infinite cycles in the following way: We add to G an ideal point, g^+, taking as neighborhoods of g^+ those open subsets of G whose closures are not compact. Then $G^+ = G \cup g^+$ is compact. Now an infinite cycle of G is defined to be a relative cycle of G^+ mod g^+. That this definition of infinite cycles is equivalent to the one used in [**2**] follows from Theorem 1.1 of [**2**].

LEMMA 1. *Given any neighborhood M of the unit element e of G, there is a neighborhood N of E such that for any infinite cycle Γ^k on H, $0 \leqq k \leqq n-1$, and for any $g \in N$, $\Gamma^k \sim g \cdot \Gamma^k$ on $M \cdot H$.*

PROOF. Let $M_{n-1} \subset M$ have a compact closure. Choose a sequence

$$M_{n-1} \supset N_{n-1} \supset M_{n-2} \supset \cdots \supset M_0 \supset N_0$$

such that N_i is obtained from M_i by the local connectedness of G in dimension i, and such that $M_i \cdot M_i \subset N_{i+1}$. Finally let N be such that $N \cdot N \subset N_0$.

Now let $g \in N$. To show that $\Gamma^k \sim g \cdot \Gamma^k$ on $M \cdot H$, it is sufficient to show that the coordinates of these cycles on the nerve of any covering U of G are homologous on $(M \cdot H)^+$. To this end, given a covering U, choose $U' <^* U$. Let X be the complement of the union of those sets of U' which contain g^+. Then X is a compact set. Let $X_1 = \overline{M}_0 \cdot X$ and $X_i = \overline{M}_{i-1} \cdot X_{i-1}$. Each X_i is a compact set.

Received by the editors December 10, 1947.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

A finite number of translations of N_{n-1} cover X_n, say

$$g_{1,n-1}\cdot N_{n-1},\ g_{2,n-1}\cdot N_{n-1},\ \cdots,\ g_{k,n-1}\cdot N_{n-1}.$$

For each i, let $U_{i,n-1}$ be a refinement of U' such that any $(n-1)$-cycle on $U_{i,n-1}$ in $g_{i,n-1}\cdot N_{n-1}$ has its projection to U' bounding in $g_{i,n-1}\cdot M_{n-1}$. Let U_{n-1} be a common refinement of these coverings.

Next, a finite number of translations of N_{n-2} cover X_{n-1}. From these we obtain a refinement U_{n-2} of U_{n-1} by the procedure above, this time using the local connectedness of G in dimension $n-2$. Proceeding in this fashion for another $n-2$ steps we arrive at a covering U_0.

Let Γ_0^k and $g\cdot\Gamma_0^k$ be the coordinates of Γ^k and $g\cdot\Gamma^k$ on U_0. We assert that $\pi\Gamma_0^k$ and $\pi g\cdot\Gamma_0^k$ are homologous on U on $(M\cdot H)^+$, where π is the projection from U_0 to U. Let Δ be the cartesian product of $|\Gamma_0^k|$ with a unit segment, subdivided simplicially in such a way that all the vertices of Δ are in the base, $\Delta_0=|\Gamma_0^k|\times 0$, and in the top, $\Delta_1=|\Gamma_0^k|\times 1$. Let $\bar{\Delta}$ be the closed subcomplex of Δ generated by those simplexes of Γ_0^k which are on X. We define a partial realization τ' of Δ on U_0 by letting $\tau'\sigma=\sigma$ if $\sigma\in\Delta_0$ and $\tau'\sigma=g\cdot\sigma$ if $\sigma\in\Delta_1$.

τ' induces a partial realization $\bar{\tau}'$ of $\bar{\Delta}$ on U_0. In view of the choices of the coverings made above, the usual argument shows that there is a full realization $\bar{\tau}$ of $\bar{\Delta}$ on U', where, if π_0 is the projection from U_0 to U', $\bar{\tau}=\pi_0\bar{\tau}'$ whenever the latter is defined. Also, $\bar{\tau}\bar{\Delta}$ is on $M_{n-1}\cdot H$.

We can now define a full realization of Δ on U in the following fashion. The projection π_0 can be so chosen that a vertex of U_0 not on X is projected into a vertex of U' which contains g^+. Since $U'<^*U$, a projection π of U' to U can be so chosen that any simplex of U_0 which has a vertex not on X is projected by $\pi\pi_0$ into the simplex of U consisting of those vertices of U which contain g^+. Any cycle in this simplex bounds in this simplex, so $\pi\pi_0\tau'(\Delta-\bar{\Delta})$ can be filled in to make a full realization of $\Delta-\bar{\Delta}$ on U, and this together with $\pi\bar{\tau}\bar{\Delta}$ makes a full realization τ of Δ on U. Since $\Gamma_0^k\times 0\sim\Gamma_0^k\times 1$ on Δ, $\tau(\Gamma_0^k\times 0)\sim\tau(\Gamma_0^k\times 1)$ on U. But $\tau(\Gamma_0^k\times 0)=\pi\pi_0\Gamma_0^k$ and $\tau(\Gamma_0^k\times 1)=\pi\pi_0 g\cdot\Gamma_0^k$. Since it is easily seen from the construction that this homology takes place on $(M\cdot H)^+$, the proof is complete.

Clearly the same proof suffices for the following lemma.

LEMMA 2. *Given any neighborhood M of e in G, there is a neighborhood N of e such that for any closed subset X of H and any cycle Γ^k of H* mod *X, $\Gamma^k\sim g\cdot\Gamma^k$ in $M\cdot H$* mod *$M\cdot X$, whenever $g\in N$.*

LEMMA 3. *If D is an open connected subset of G, then any two points* *of D lie on a compact connected subset of D.*

PROOF. Since G is lc^0, any neighborhood O of a point d of D contains a neighborhood W of d with $W\subset D$ such that any point $w\in W$ lies, together with d, on a compact continuum in $O\cap D$. Now let D_1 be the set of all points of D which can be joined to a fixed point $d_1\in D$ by compact continua. Then, by the above, D_1 is both open and closed in D. Hence, since D is connected, D_1 is all of D.

LEMMA 4. *If O is a neighborhood of e such that $\mathcal{C}(O\cdot H)$ (where $\mathcal{C}$ means closure) is not all of G, then $O\cdot H-H$ has at least two components.*

PROOF. Let g be a point of $G-\mathcal{C}(O\cdot H)$, and let K be a compact connected set which contains both e and g. Let N be a neighborhood of e in O, chosen by Lemma 1. A finite number of translations of N cover K, and from these we may choose a sequence

$$e\in N,\ N_1,\ N_2,\ \cdots,\ N_k\ni g$$

where $N_i=g_i\cdot N$ and such that $N_i\cap N_{i+1}\neq 0$. Let $\bar{g}_i$ be a point of $N_i\cap N_{i+1}$. Now, $\bar{g}_{i-1}\in g_i\cdot N$, so $g_i^{-1}\cdot\bar{g}_{i-1}\in N$. Hence, by Lemma 1,

$$\Gamma^{n-1}\sim g_i^{-1}\cdot\bar{g}_{i-1}\cdot\Gamma^{n-1}$$

where Γ^{n-1} is a nonbounding cycle on H. Therefore,

$$g_i\cdot\Gamma^{n-1}\sim\bar{g}_{i-1}\cdot\Gamma^{n-1}.$$

Similarly, $\bar{g}_i\in g_i\cdot N$, and

$$g_i\cdot\Gamma^{n-1}\sim\bar{g}_i\cdot\Gamma^{n-1}.$$

Thus, we have

$$\Gamma^{n-1}\sim g\cdot\Gamma^{n-1}.$$

Now $\Gamma^{n-1}-g\cdot\Gamma^{n-1}$ is a cycle of $H\cup(G-O\cdot H)$ and $\Gamma^{n-1}-g\cdot\Gamma^{n-1}\sim 0$ in G. Hence [**3**, p. 227 (14.2)] there is in G a cycle Γ^n mod $(H\cup(G-O\cdot H))$ such that $F\Gamma^n=\Gamma^{n-1}-g\cdot\Gamma^{n-1}$. Let $\mathbf{\Gamma}^n$ be the fundamental n-cycle of G, and let $\mathbf{\Gamma}_1^n$ be the part of $\mathbf{\Gamma}^n$ on $G-(O\cdot H-H)$. Let $\mathbf{\Gamma}_2^n=\mathbf{\Gamma}^n-\mathbf{\Gamma}_1^n$.

In a neighborhood of any point of $O\cdot H-H$, Γ^n is homologous to some multiple of $\mathbf{\Gamma}^n$. If we assume that $O\cdot H-H$ is connected, then (cf. [**1**, p. 569]) this multiple, is the same for all points of $O\cdot H-H$, that is, $r\Gamma^n=\mathbf{\Gamma}_2^n$.

By definition, $\mathbf{\Gamma}_1^n$ is on $H\cup(G-O\cdot H)$. Since H and $G-O\cdot H$ are closed and disjoint, and since dim $H<n$, $\mathbf{\Gamma}_1^n$ must be on $G-O\cdot H$, so $F\mathbf{\Gamma}_1^n$ is also on $G-O\cdot H$.

But, from $0=F\overline{\Gamma}^n=F(\overline{\Gamma}_1^n+\overline{\Gamma}_2^n)$, we have

$$F\overline{\Gamma}_1^n = -F(\overline{\Gamma}_2^n) = -F(r\Gamma^n) = -r(\Gamma^{n-1} - g\cdot\Gamma^{n-1}).$$

This is not on $G-O\cdot H$, since Γ^{n-1} is on H. Thus, the assumption that $O\cdot H-H$ has only one component leads to a contradiction.

We now choose a fixed connected neighborhood of e, satisfying the condition of Lemma 4, and denote by J the product of H by this neighborhood. We note that J is a connected generalized n-manifold. It is not a group, but for any two elements of J which are close enough to H, their product in G is in J.

LEMMA 5. *H is the boundary of each domain of $J-H$.*

PROOF. Let D be any component of $J-H$. Since J is lc^0, D is open. Some point $h\in H$ is a limit point of D, or else J would not be connected. Let O be a neighborhood of e. Then $h\cdot O$ contains a point $d\in D$, that is, $h\cdot o=d$. Now $h^{-1}\cdot d=o\in O$. But $h^{-1}\cdot d$ is also in D. For $o\notin H$, and, since H is connected, $H\cdot o$ lies in one component of $J-H$. Since $h\cdot o=d$ is in D, $H\cdot o$ lies in D, and consequently, $e\cdot o=o$ is in D. Therefore e is a limit point of D. Similarly, if $\bar{h}$ is any other point of H, then the neighborhood $\bar{h}\cdot O$ contains $\bar{h}\cdot o$ which is in $H\cdot o$ and therefore in D. Thus, $\bar{h}$ is a limit point of D, which proves the lemma.

LEMMA 6. *$J-H$ has just two components.*

PROOF.[2] By Lemma 2, it is enough to show that H does not have three complementary domains. Suppose there were three, D_0, D_1 and D_2. Let p_1, p_2 be points in D_1 and D_2 respectively, and let Y_1, Y_2 be neighborhoods of p_1, p_2 such that $\overline{Y}_i$ is compact and is in D_i.

$\gamma^0=p_1-p_2$ is a compact 0-dimensional cycle in $Y=Y_1\cup Y_2$. γ^0 not ~ 0 in $J-H$, since p_1 and p_2 are in different components. But for any point $d_0\in D_0, \gamma^0\sim 0$ in $J-d_0\cdot H$. For let O be a neighborhood of e not meeting $d_0\cdot H$, which is in D_0, and let O' be chosen so that every compact 0-cycle in O' bounds in O. Choose $d_1\in O'\cap D_1$ and $d_2\in O'\cap D_2$. Then $d_1\sim d_2$ in O. By Lemma 3, $p_1\sim d_1$ in D_1, $p_2\sim d_2$ in D_2. Hence, $p_1\sim p_2$ in $D_1\cup D_2\cup O$, which does not meet $d_0\cdot H$.

Now, by Lemma 5.2 of [2], there is a compact cocycle γ_n in Y such that $(\overline{\Gamma}^n\cdot\gamma_n)\sim\gamma^0$ in $J-H$, where $\overline{\Gamma}^n$ is the fundamental n-cycle of J, and such that $\gamma_n\sim 0$ in $J-d_0\cdot H$, for any $d_0\in D_0$. Since γ_n is a compact cocycle of $D_1\cup D_2$, there is an infinite n-cycle Γ^n of $D_1\cup D_2$

[2] The main outline of this proof, and to some extent that of Lemma 7, is derived from Wilder [6, 7].

such that $\mathrm{KI}(\Gamma^n \cdot \gamma_n) = 1$. Let $\Gamma^{n-1} = F\Gamma^n$, so that Γ^{n-1} is an infinite cycle of H.

We now choose a neighborhood M of e which does not meet $\overline{Y}$, and a neighborhood N satisfying the conditions of Lemma 1. Let $d_0 \in D_0 \cap N$. Then $\Gamma^{n-1} \sim d_0 \cdot \Gamma^{n-1}$ in $M \cdot H$. Let $\Gamma^n = \{\Gamma_\zeta^n\}$ and let the chains involved in the homology $\Gamma^{n-1} \sim d_0 \cdot \Gamma^{n-1}$ be $\{C_\zeta^n\}$. Then $\{\Gamma_\zeta'^n\} = \{\Gamma_\zeta^n - C_\zeta^n\}$ is such that $F\Gamma_\zeta'^n = d_0 \cdot \Gamma_\zeta^{n-1}$. By construction, $\mathrm{KI}(\Gamma_\zeta'^n \cdot \gamma_n^\zeta) = \mathrm{KI}(\Gamma_\zeta^n \cdot \gamma_n^\zeta)$, since none of the chains C_ζ^n meet $\overline{Y}$.

$\{\Gamma_\zeta'^n\}$ is not necessarily a Čech cycle. But, for each covering U_ζ, let $U_{\rho(\zeta)}$ be an essential refinement (see [3, II 27: 13]) of U_ζ relative to cycles of J^+ mod $(d_0 \cdot H)^+$. Then $\{\Gamma_\zeta''^n\} = \{\pi_{\rho(\zeta)}^\zeta \Gamma_{\rho(\zeta)}'^n\}$ is a Čech cycle mod $(d_0 \cdot H)^+$ and $\mathrm{KI}(\Gamma''^n_\zeta \cdot \gamma_n^\zeta) = \mathrm{KI}(\Gamma_\zeta'^n \cdot \gamma_n)$ for all ζ.

But now we have reached a contradiction. For $\gamma_n \sim 0$ in $J - d_0 \cdot H$, so its Kronecker index with any infinite n-cycle of $J - d_0 \cdot H$ must be zero. But $\mathrm{KI}(\Gamma''^n \cdot \gamma_n) = \mathrm{KI}(\Gamma^n \cdot \gamma_n) = 1$.

LEMMA 7. *For each point $h \in H$, $r^k(J-H, h) = 0$ for $1 \leqq k \leqq n-1$ and $r^0(J-H, h) = 1$.*

PROOF. It is sufficient to consider the case $h = e$. Given any neighborhood V of e, choose a neighborhood V_1 such that $\mathcal{C}(\mathcal{C}(V_1 \cdot V_1) \cdot B(V))$ does not meet $\overline{V}_1$, where $B(V)$ is the boundary of V. Next choose a neighborhood V_2 such that if $\gamma^0 \subset V_2$, then $\gamma^0 \sim 0$ in V_1. Let V_3 be such that $\overline{V}_3 \cdot H$ does not contain all of V_2, and, finally, let W be such that if $\gamma^k \subset W$, then $\gamma^k \sim 0$ in V_3. We assert that for $k \geqq 1$, any γ^k in $W - H$ bounds in $V - H$.

For let A and B be the two components of $J - H$ and let $\gamma^k = \gamma_a^k + \gamma_b^k$, where γ_a^k is the part of γ^k in A. Since $k \geqq 1$, γ_a^k is a cycle and it is sufficient to show that $\gamma_a^k \sim 0$ in $V \cap A$. If it does not, let O be an open set in $W \cap A$ such that γ^k is in O and O does not meet H. Then, by Lemma 5.2 of [2], there is a compact cocycle γ_{n-k} in O such that $\overline{\Gamma}^n \cdot \gamma_{n-k} \sim \gamma^k$ in O, γ_{n-k} not ~ 0 in $V \cap A$, and $\gamma_{n-k} \sim 0$ in V_3. Let Γ^{n-k} be an infinite cycle of $V \cap A$ such that $\mathrm{KI}(\Gamma^{n-k} \cdot \gamma_{n-k}) = 1$.

In order to apply an argument similar to that of the preceding lemma, we choose a point of B in the following fashion. Let c be a point of B in V_2 and not in $\mathcal{C}(V_3 \cdot H)$. By the choice of V_2, there is a continuum K in V_1 which contains both c and e. Let M be a neighborhood of e such that $M \cdot H$ does not meet $\overline{O}$. Choose N by Lemma 1 and so that $N \cdot K$ is in V_1. $N \cdot K - H$ is an open subset of J and hence is locally connected. Consequently, each component of $N \cdot K - H$ is also open. Let C be that component which contains c. Since C is open and $N \cdot K$ is connected, some point h in H must be a limit point of C. h is in $\overline{V}_1$, since $N\cdot K$ is in V_1, and therefore $D=C\cdot h^{-1}$, containing $d=ch^{-1}$, is an open connected subset of B and e is a limit point of D. Also, since c is not in $V_3\cdot H$, neither is d.

From $N\cdot D$ and N itself a simple chain of regions running from e to d can be extracted, each element of the chain being a translation of N.

Returning now to Γ^{n-k}, let Γ^{n-k-1} be the part of $F\Gamma^{n-k}$ on H, so that Γ^{n-k-1} is a cycle of H mod X, where $X=\overline{V}\cap\mathcal{C}(H-V)$. Using the simple chain above, we have $\Gamma^{n-k-1}\sim d\cdot\Gamma^{n-k-1}$ in $M\cdot D$. Let the chains of this homology be $\{C_\zeta^{n-k}\}$. Then $\Gamma_\zeta^{n-k}-C_\zeta^{n-k}$, for each ζ, is, by the choice of V_1, an infinite cycle of V_3. Also, by the choice of M, no C_ζ^{n-k} meets $\overline{O}$, so $\mathrm{KI}((\Gamma_\zeta^{n-k}-C_\zeta^{n-k})\cdot\gamma_{n-k}^\zeta)=\mathrm{KI}(\Gamma_\zeta^{n-k}\cdot\gamma_{n-k}^\zeta)=1$ for each ζ.

Now we can proceed to the same contradiction we reached in the previous lemma, since $\gamma_{n-k}\sim 0$ in V_3 so its Kronecker index with any infinite cycle of V_3 is zero. This disposes of the case $k\geqq 1$.

For $k=0$, let $\bar{\gamma}^0$ be based on a pair of points, one in $W\cap A$ and the other in $W\cap B$. The proof used above applies to show that any γ^0 in $W-H$ is homologous in $V-H$ to a multiple of $\bar{\gamma}^0$.

LEMMA 8. *For each point h of H, $r_k(h)=0$ for $k<n-1$ and $r_{n-1}(h)=1$.*

This is an immediate consequence of Theorem 6.2 of [2] and Lemma 7.

LEMMA 9. *H is lc^{n-1}.*

PROOF. Given a neighborhood V of e, choose V_1 in V such that any γ^{k+1} on $\overline{V}_1$ bounds in V. Choose $W\subset V_1$ by Lemma 7 so that any γ^k in $A\cap W$ bounds in $A\cap V_1$ and similarly for B. We assert that any γ^k in $W\cap H$ bounds on $\overline{V}\cap H$.

To show this it is enough to show that for any neighborhood O of e, $\gamma^k\sim 0$ in $(O\cdot H)\cap\overline{V}$. In turn, to prove this it is sufficient to show that given any such γ^k and O, and given any covering U_1, then there is a refinement U_2 such that $\pi_2^1\gamma_2^k\sim 0$ in $(O\cdot H)\cap\overline{V}$.

By Lemma 1 we can choose a point $a\in A\cap O$ such that $\gamma^k\sim a\cdot\gamma^k$ in $O\cdot(W\cap H)$ and we can choose a similar point b in $B\cap O$. By the choice of W, $a\cdot\gamma^k\sim 0$ in $A\cap V_1$, and similarly for $b\cdot\gamma^k$. Thus, we have families of chains $\{C_{a,\zeta}^{k+1}\}$ and $\{C_{b,\zeta}^{k+1}\}$ in $O\cdot(W\cap H)$, $\{D_{a,\zeta}^{k+1}\}$ in $A\cap V_1$ and $\{D_{b,\zeta}^{k+1}\}$ in $B\cap V_1$ such that

$$FC_{a,\zeta}^{k+1} = a\cdot\gamma_\zeta^k - \gamma_\zeta^k, \qquad FC_{b,\zeta}^{k+1} = b\cdot\gamma_\zeta^k - \gamma_\zeta^k,$$
$$FD_{a,\zeta}^{k+1} = a\cdot\gamma_\zeta^k, \qquad FD_{b,\zeta}^{k+1} = b\cdot\gamma_\zeta^k.$$

Hence, for each ζ, $D_{a,\zeta}^{k+1}-C_{a,\zeta}^{k+1}+C_{b,\zeta}^{k+1}-D_{b,\zeta}^{k+1}$ is a cycle δ_ζ^{k+1} on U_ζ in

V_1. There is a refinement U_2 of U_1 such that $\pi_2^1\delta_2^{k+1}$ is the coordinate of a Čech cycle, δ^{k+1} on $\overline{V}_1$. By the choice of V_1, $\delta^{k+1}\sim 0$ in V, so there is a chain E^{k+2} on U_1 such that

$$FE^{k+2} = \pi_2^1\delta_2^{k+2}.$$

Let $E^{k+2}=E_a^{k+2}+E_b^{k+2}$, where E_b^{k+2} is the part of E^{k+2} on $\overline{B}$. Now

$$FE_a^{k+2} - \pi_2^1 D_{a,2}^{k+1} = - C_{a,2}^{k+1} + C_{b,2}^{k+1} - D_{b,2}^{k+1} - FE_b^{k+2}.$$

The chain on the right-hand side is in $O\cdot B$ while that on the left is on $\overline{A}$. Hence, since $\overline{A}\cap\overline{B}=H$, $E^{k+1}=FE_a^{k+2}-\pi_2^1D_{a,2}^{k+1}$ is in $O\cdot H$ and, of course, in V. But

$$F(- E^{k+1}) - F(\pi_2^1 D_{a,2}^{k+1}) = \pi_2^1 a\cdot\gamma_a^k.$$

Hence, $\pi_2^1 a\cdot\gamma_2^k\sim 0$ in $(O\cdot H)\cap V$. But $a\cdot\gamma_2^k\sim\gamma_2^k$ in $O\cdot(W\cap H)$, so $\pi_2^1\gamma_2^k\sim 0$ in $(O\cdot H)\cap V$.

At this point, we have shown, by Lemmas 8 and 9, that H has the local properties of a generalized manifold. To complete the proof it only remains to show that H is orientable, that is, that it carries an $(n-1)$-cycle which is not carried by any proper closed subset of H.

By Lemma 8, there are neighborhoods O_1 and O_2 of e such that there is an $(n-1)$-cycle mod $H-O_1$ which does not bound mod $H-O_2$. By group translation, every point of H has associated with it such a non-bounding relative $(n-1)$-cycle. Now an argument due to Smith [5] shows that we can carry through in the present situation the proof of Theorem 7.1 of [1] to obtain the desired $(n-1)$-cycle.

In conclusion, we point out that by restricting G, we can lighten the hypothesis on H.

THEOREM. *Let G be a locally compact separable metric topological group which is also an orientable n-dimensional generalized manifold. Let H be a closed connected $(n-1)$-dimensional subgroup. Then H is an orientable generalized manifold if any one of the following conditions is satisfied:*

(1) *H separates some open set of G.*

(2) *For some open set O of H, there is a nonbounding $(n-1)$-cycle of H mod $H-O$.*

(3) *G is locally euclidean.*

The Pontrjagin duality theorem for case (3) and Theorem 6.5 of [2] for case (2) show that both (3) and (2) imply (1). Now the proof of Lemma 1 of [4] shows that (1) yields a neighborhood of H which is separated by H, that is, our Lemma 4. Since this is the only place in our proof where the original hypothesis on H is used, the rest of the proof can remain unchanged.

In case (3), if dim $G=3$, we have Montgomery's theorem, for any 2-dimensional generalized manifold is locally euclidean [8].

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