ON POLYNOMIALS AND LAGRANGE'S FORM OF THE GENERAL MEAN-VALUE THEOREM

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Suppose that in $(a<x<b)$ (hereafter referred to as (a, b)),

(1) $f(x)$ is defined and has derivatives of the first n orders.

Then, from the general mean-value theorem with Lagrange's form of remainder follows the existence of $\theta=\theta(x, h)$, such that

$$f(x+h) = f(x) + \sum_{r=1}^{n-1} \frac{h^r}{r!} f^{(r)}(x) + \frac{h^n}{n!} f^{(n)}(x+\theta h) \tag{2}$$

$$\text{for } a < x < x + h < b.$$

The θ in (2) is sometimes a uniquely determinate function of x and h in the relevant domain $a<x<x+h<b$ (hereafter referred to as R), as, for instance, if $f^{(n+1)}(x)$ exists and is not zero in (a, b). If, further, $f^{(n+1)}(x)$ is continuous in (a, b), it is easily seen that

$$\lim_{h\to+0} \theta(x, h) = \frac{1}{n+1} \qquad \text{in } a < x < b.$$

It is also possible for $\theta(x, h)$ to be an analytic function, for example,

$$\theta(x, h) = h^{-1} \log\left(1 + \sum_{r=1}^{\infty} \frac{h^r\Gamma(n+1)}{\Gamma(n+r+1)}\right),$$

which happens when $f(x)=e^x$.

It would, therefore, seem worth while to determine the types of functions that are or are not possible for $\theta(x, h)$. Inquiry in this direction has led to the results of this paper, namely:

THEOREM 1. *If a polynomial $\theta(x, h)$ exists such that* (2) *is true with $\theta(x, h)$ in place of θ, then $f^{(n+1)}(x)$ exists in (a, b) and either*

$$f^{(n+1)}(x) = 0 \qquad \textit{in } (a, b) \tag{a}$$

or

(b) $f^{(n+1)}(x)$ *= a constant* $\neq 0$ *in* (a, b), *and* $\theta(x, h)$ *is uniquely determinate and equal to* $1/(n+1)$ *in* R.

THEOREM 2. *If* (2) *is true with* $\theta(x, h)=c(x)+h^d\phi(x, h)$ *where*

(3) $\phi(x, h)$ *is bounded in* R;

(4) d *is a constant greater than* 1;

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(5) *$\partial\theta/\partial x$, $\partial^2\theta/\partial^2 x$ are continuous in x, and θ is bounded in R;*
(6) *for all sufficiently small h, $1+h(\partial\theta/\partial x)\neq 0$ in R;*
then, also, (a) *and* (b) *of Theorem* 1 *are true.*

It is significant that, if θ is uniquely determined by (2) in R and not equal to $1/(n+1)$, then $\theta=\theta(x, h)$ cannot be equal to a polynomial in R (by Theorem 1) or even to an analytic function (by Theorem 2) satisfying

(7) $\lim_{h\to 0}\partial\theta/\partial h=0$ for every x in (a, b).

[In the following we write $\theta(x, 0)$ for $\lim_{h\to+0}\theta(x, h)$ and $\theta_1(x, 0)$ for $\lim_{h\to+0}(\theta(x, h)-\theta(x, 0)/h)$ (which limits obviously exist in the contexts of the two theorems), and θ_{rs} for $(\partial^{r+s}/\partial x^r\partial h^s)\theta$, wherever the latter obviously exists.]

PROOF OF THEOREM 1.

(8) The conditions (5) and (6) are obviously satisfied here and (2) is true by hypothesis.

On account of the consequent boundedness of θ in R, and the continuity of θ in x, follows

(9) $y=x+\theta h$ for every y in (a, b), with any sufficiently small h and a correspondingly chosen x such that (x, h) lies in R.

From (8) and (9) follows

(10) $f^{n+1}(x)$ and $f^{n+2}(x)$ exist and are continuous in (a, b).

Now, from the general mean-value theorem follows

$$
\begin{aligned}
f(x+h) &= f(x)+\sum_{r=1}^{n}\frac{h^r}{r!}f^{(r)}(x)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x) \\
&\quad+\frac{h^{n+2}}{(n+2)!}f^{(n+2)}(x+\theta_1 h),\quad 0<\theta_1<1,\quad (x, h)\subset R;
\end{aligned}
\tag{11}
$$

and from (2) and the same theorem applied to $f^n(x+\theta h)$ follows

$$
\begin{aligned}
f(x+h) &= f(x)+\sum_{r=1}^{n}\frac{h^r}{r!}f^{(r)}(x)+\frac{h^{n+1}\theta}{(n)!}f^{(n+1)}(x) \\
&\quad+\frac{h^{n+2}\theta^2}{n!2!}f^{(n+2)}(x+\theta_2\theta h),\qquad 0<\theta_2<1,\ (x, h)\subset R.
\end{aligned}
\tag{12}
$$

Subtracting (12) from (11) and making $h\to+0$ after division by h^{n+1}, it follows by (10) that

$$
f^{(n+1)}(x)\left[1-(n+1)\theta(x, 0)\right]=0. \tag{13}
$$

Using (13) in (11) and (12), and making $h\to+0$ after division of their difference by h^{n+2}, it follows, again by (10),

$$(14)\qquad f^{(n+2)}(x)\left[1-\frac{(n+1)(n+2)}{2}\theta^2(x,0)\right] - f^{(n+1)}(x)(n+1)(n+2)\theta_1(x,0)=0.$$

Now, either

(15a) $f^{(n+1)}(x)=0$ everywhere in (a, b);

or

(15b) $f^{(n+1)}(x)\neq 0$ everywhere in (a, b);

or

(15c) on account of the continuity (by (10)) of $f^{(n+1)}(x)$ there exists a closed interval (a_1, b_1) contained in (a, b) such that $f^{(n+1)}(x)\neq 0$ for $a_1<x<b_1$, and one at least of $f^{(n+1)}(a_1)$ and $f^{(n+1)}(b_1)$ is zero.

If (15c) were possible, then we should have, by (13) and (14),

$$f^{(n+2)}(x)\cdot n/2(n+1) - f^{(n+1)}(x)(n+1)(n+2)\theta_1(x,0)=0 \quad \text{in } (a_1<x<b_1),$$

and hence $f^{(n+1)}(x)=A\cdot\exp\{\phi(x)\}$ in $a_1<x<b_1$, where $\phi(x)$ is a polynomial and A is a constant, and making $x\rightarrow a_1$ or b_1 in this, there would follow that $f^{(n+1)}(x)=0$ in $a_1<x<b_1$, which contradicts (15c). Hence

(16) (15c) is impossible, and $f^{(n+1)}(x)=A\exp\{\phi(x)\}$ in $a<x<b$, where $\phi(x)$ is a polynomial and $A=$ a constant $\neq 0$, if $f^{(n+1)}(x)\neq 0$ for some x in (a, b).

Now differentiating (2) with respect to x and h, as is obviously permissible on account of (10), and subtracting, and dividing by h^{n-1}, it follows that

$$f^{(n)}(x)-f^{(n)}(x+\theta h)=\frac{h}{n}f^{(n+1)}(x+\theta h)[\theta-1+h\theta_{01}-h\theta_{10}] \quad \text{in } R.$$

Differentiating this (possible by (10)) with respect to x and using (16) we get

(17) $\exp\{k(x, h)\}=g(x, h)$ in R, in case (15b), where $k(x, h)=\phi(x)-\phi(x+\theta h)$ and $k(x, h)$ and $g(x, h)$ are polynomials in x and h.

It is now seen by the theory of analytic continuation that (17) is impossible unless $k(x, h)$ is a constant, which again is seen to be zero by keeping x fixed and making $h\rightarrow +0$. Hence

$$(18)\qquad \phi(x)=\phi(x+\theta h) \quad \text{in } R.$$

Now from (2) obviously follows

(19) $f(x)$ is a polynomial of degree not greater than n in (a, b) if $\theta(x, h)\equiv 0$. Also, by continuous variation of x and h in R it follows from (18) that

(20) $\phi(x) =$ a constant k in (a, b) if $\theta(x, h) \not\equiv 0$,
and hence, using (16), follows

(21) $f^{(n+1)}(x) = Ae^k$ in (a, b) where $A \neq 0$ if $f^{(n+1)}(x) \neq 0$ in (a, b),
Now the theorem follows from (10), (15a), (15b), (16), (19) and (21). since, when $f^{(n+1)}(x) =$ a constant $\neq 0$, $\theta = 1/(n+1)$ and is uniquely determined by (2) in R.

PROOF OF THEOREM 2. In this case, the statements (8) to (14) follow as above, and $\theta_1(x, 0) = 0$ since $d > 1$. Hence (13) and (14) now become

$$f^{(n+1)}(x)\left[1 - (n+1)c(x)\right] = 0 \qquad \text{in } (a, b), \tag{22}$$

$$f^{(n+2)}(x)\left[1 - \frac{(n+1)(n+2)}{2} c^2(x)\right] = 0 \qquad \text{in } (a, b). \tag{23}$$

Hence either

(24a) $f^{(n+1)}(x) = 0$ every where in (a, b),

or

(24b) $f^{(n+1)}(x) = c \neq 0$ for some x in (a, b).

Then, (22) and (23) give

(25) $c(x) = 1/(n+1)$ wherever $f^{(n+1)}(x) \neq 0$,

(26) $f^{(n+2)}(x) = 0$ wherever $f^{(n+1)}(x) \neq 0$.

The theorem now follows from (10), (24a), (24b), (25) and (26), since, when $f^{(n+1)}(x) = c \neq 0$ in (a, b), θ in (2) is uniquely determined in R.

Note added January 18, 1948. The conclusions (a) and (b) of Theorem 1 are true if (2) holds with $\theta(x, h)$ in place of θ, where

$$\theta(x, h) = \sum_{r=0}^{m} h^r \theta_r(x),$$

and $\theta_1(x)$ is a polynomial, $\theta(x, h)$ satisfies (6) and each of the functions $\theta_\nu(x)$ satisfies (5). The line of proof is briefly as follows:

The arguments up to and including (16) are the same as above, and the equation in (17) is now true with $K(x, h) = \phi(x) - \phi(x + \theta h)$, and $K(x, h)$ and $g(x, h)$ polynomials in h for fixed x. The rest of the argument is the same as before.

The conclusion (20) can also be seen directly as follows: Differentiating (18) with respect to h, we have

$$\phi'(x + \theta h)\left(\theta + h\frac{\partial \theta}{\partial h}\right) = 0.$$

Making $h \to 0$ in this and noting that $\theta(x, 0) = 1/(n+1)$ in case (15b) we have $\phi'(x) = 0$, and hence $\phi(x) = k$, a constant in (a, b).

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