GENERALIZATION OF MENGER'S RESULT ON THE STRUCTURE OF LOGICAL FORMULAS

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Menger's paper[2] gives necessary and sufficient conditions that an expression containing sentential variables and unary and binary sentential connectives be a formula in the Łukasiewicz notation. This paper extends his result to expressions containing n-ary symbols for all $n \geqq 0$. Sentential variables and constants are treated as the case $n=0$.

An expression is a sequence $s_1 \cdots s_k$ such that s_i for $i=1, \cdots, k$ is an n-ary symbol for some n. An initial segment of such an expression is an expression $s_1 \cdots s_i$ where $i<k$; a terminal segment is an expression $s_t \cdots s_k$ where $t>1$. A formula is a sequence $sz_1 \cdots z_n$ where s is an n-ary symbol, and $z_1, \cdots, z_n$ are formulas. The measure $[s]$ of an n-ary symbol s is $n-1$. The measure $[s_1 \cdots s_k]$ of an expression $s_1 \cdots s_k$ is $[s_1]+ \cdots +[s_k]$.

THEOREM. *Necessary and sufficient conditions that an expression* $x=s_1 \cdots s_k$ *be a formula are*:

$$[y] \geqq 0 \text{ for each initial segment } y \text{ of } x, \tag{1}$$

and

$$[x] = -1. \tag{2}$$

PROOF. Suppose s is an n-ary symbol, $z_1, \cdots, z_h$, $h \geqq 0$ are formulas, z is an initial segment of a formula z_{h+1}, and $z_1, \cdots, z_{h+1}$ satisfy (1) and (2). Then

$$\begin{aligned} [sz_1 \cdots z_h] &= [s] + [z_1] + \cdots + [z_h] \\ &= (n-1) - 1 - \cdots - 1 = n - 1 - h \end{aligned} \tag{3}$$

and

$$[sz_1 \cdots z_h z] = n - 1 - h + [z] \geqq n - 1 - h. \tag{4}$$

PROOF OF NECESSITY. Let $x=sz_1 \cdots z_n$ be a formula where by the induction hypothesis s is an n-ary symbol and $z_1, \cdots, z_n$ are formulas

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[2] Menger, Karl, *Eine elementare Bemerkung über die Struktur logischer Formeln*, Ergebnisse eines mathematischen Kolloquiums, vol. 3, 1930–1931, pp. 22–23.

satisfying (1) and (2). An initial segment y of x has one of the forms (3) or (4), with $h<n$ so that $[y]\geqq 0$, and (3) applies to x with $h=n$, giving $[x]=-1$.

PROOF OF SUFFICIENCY. An expression of length one is a formula by (2) and the definition. Suppose all expressions of length less than k satisfying (1) and (2) are formulas, and let $x=s_1 \cdots s_k$ be an expression of length $k>1$ satisfying (1) and (2).

LEMMA. *Starting at any symbol s_i of x, $1\leqq i\leqq k$, there is a unique segment satisfying* (1) *and* (2).

PROOF. For any terminal segment $s_t \cdots s_k$ of x, we have $[s_1 \cdots s_{t-1}]+[s_t \cdots s_k]=[x]$ or $[s_t \cdots s_k]\leqq -1$ by (1) and (2). Thus, for any symbol s_i, $1\leqq i\leqq k$, there is a symbol s_j, $i\leqq j\leqq k$, such that $[s_i \cdots s_j]=-1$. For each integer i, only the smallest such integer j provides a segment $s_i \cdots s_j$ satisfying (1) as well as (2).

By (1), s_1 is a connective s for some $n>0$. The lemma may be applied, starting at s_2, to exhaust the symbols of x by constructing consecutive segments $z_1, \cdots, z_h$, each a formula by the induction hypothesis. Now for $x=sz_1 \cdots z_h$, by (2) and (3), $[x]=-1=n-1-h$ or $h=n$; hence x is a formula.

This completes the proof of the theorem.

COROLLARY. *In any formula, starting at a given symbol, there is a unique consecutive part which is a formula.*

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