THE REMAINDER IN APPROXIMATIONS BY MOVING AVERAGES

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1. Introduction. Many of the processes of interpolation or smoothing are of the following sort. A function $L(s)$, defined for all real s, characterizes the process. Given a function $x(s)$, the function

$$y(t) = \sum_{j=-\infty}^{\infty} x(j)L(t-j) \tag{1}$$

is constructed, when possible; $y(t)$ is thought of as an approximation of $x(t)$. The remainder in the approximation is

$$R[x] = x(t) - y(t). \tag{2}$$

In the conventional processes of smoothing or interpolation, $L(s)$ is a function which vanishes for all $|s|$ sufficiently large. I. J. Schoenberg[2] has recently introduced a class of formulas (1), (2) in which $L(s)$ is an analytic function and the series (1) does not consist of a finite number of terms.

Schoenberg gives an elegant criterion for recognizing cases in which the approximating process is exact for polynomials of degree $n-1$; that is, cases in which $R[x]=0$, for all t, whenever $x(s)$ is a polynomial of degree $n-1$.[3] In the present paper we obtain an integral representation of such operations $R[x]$ in terms of the nth derivative $x^{(n)}(s)$. The representation is precisely of the sort that holds when $R[x]$ is a linear functional on certain spaces of functions $x(s)$ defined on a *finite* s-interval.

2. The integral representation. We shall consider an operation which is more general than (1), (2). *Let $g(s, t)$ be a function which, for each number t in a given set $\mathfrak{T}$, is of bounded variation in s on each finite s-interval.* Given any function $x(s)$, put

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[2] *Contributions to the problem of approximation of equidistant data by analytic functions*, Quarterly of Applied Mathematics vol. 4 (1946) pp. 45–99 and 112–141.

[3] Loc. cit. Theorem 2B, p. 64. Schoenberg's criterion is valid whether $L(s)$ is a symmetric function or not.

Throughout the present paper "polynomial of degree k" is to be understood as "polynomial of proper degree k or less."

$$y(t) = \int_{-\infty}^{\infty} x(s) d_s g(s, t), \tag{3}$$

and

$$R[x] = x(t) - y(t), \qquad t \in \mathfrak{T}.$$

Unless the contrary is stated, *integrals on infinite ranges are to be understood either as Lebesgue-Stieltjes integrals or as improper Lebesgue-Stieltjes integrals*, that is, limits of integrals over finite intervals as the intervals become infinite. Either convention may be adopted, providing that it is consistently held. We shall say that $R[x]$ exists if $y(t)$ and $x(t)$ exist and are finite for each $t \in \mathfrak{T}$.

The integral (3) reduces to the sum (1) if $g(s, t)$ is, for each $t \in \mathfrak{T}$, constant on each interval $j < s < j+1$ and if $g(j+0, t) - g(j-0, t) = L(t-j), j = \cdots, -2, -1, 0, 1, \cdots$. The name "moving average" is most appropriate to (3) when $d_s g(s-m, t) = d_s g(s, t+m)$ for all numbers m or for all integers m; we do not require that g satisfy this condition.

Assume that $R[x]$ *exists and vanishes, for all* $t \in \mathfrak{T}$, *whenever* $x(s)$ *is a polynomial of degree* $n-1$ ($n \geqq 1$). Put

$$p(s, s') = (s - s')^{n-1}/(n-1)!;$$

$$\psi_{s'} = \psi_{s'}(s) = \begin{cases} 0 & \text{if } s \leqq s', \\ p(s, s') & \text{if } s > s'. \end{cases} \tag{4}$$

For each fixed s', $R[\psi_{s'}]$ exists, since $\psi_{s'}$ is a truncated polynomial of degree $n-1$. Hence the function $k(s', t) = R[\psi_{s'}]$ is defined for all s' and all $t \in \mathfrak{T}$. An alternative formula for $k(s', t)$ is the following:

$$k(s', t) = \begin{cases} \int_{-\infty}^{s'} p(s, s') dg(s) & \text{if } s' < t, \\ -\int_{s'}^{\infty} p(s, s') dg(s) & \text{if } s' \geqq t, \end{cases} \qquad t \in \mathfrak{T}; \tag{5}$$

here[4] and elsewhere $dg(s)$ *is to be understood as an abbreviation for* $d_s g(s, t)$. To establish (5), observe that, if $s' \geqq t$, $\psi_{s'}(t) = 0$, and

$$R[\psi_{s'}] = \psi_{s'}(t) - \int_{-\infty}^{\infty} \psi_{s'}(s) dg(s) = -\int_{s'}^{\infty} p(s, s') dg(s),$$

by (4). The other part of (5) is derived similarly, with the use of the

[4] Whether the value s' is included or excluded in the range of integration of these integrals is immaterial, since $p(s', s') = 0$.

additional fact that

$$R[p(s, s')] = 0 = p(t, s') - \int_{-\infty}^{\infty} p(s, s')dg(s), \qquad t \in \mathfrak{T}. \tag{6}$$

This relation is true because $p(s, s')$ is a polynomial of degree $n-1$, for each s'.

Suppose that $x(s)$ is a function whose derivative of order $n-1$ exists and is absolutely continuous on every finite s-interval. Put

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dg(s) \int_0^s p(s, s')x^{(n)}(s')ds', \\ R^*[x] &= \int_{-\infty}^{\infty} x^{(n)}(s')k(s', t)ds', \end{aligned} \qquad t \in \mathfrak{T}.$$

THEOREM. *A necessary and sufficient condition that $R[x]$ and $R^*[x]$ exist and be equal is that I exist and that the order of integration in I be invertible, for all $t \in \mathfrak{T}$.*

PROOF. For brevity put

$$z = p(s, s')x^{(n)}(s').$$

Sufficiency: Since the order of integration in I is invertible.

$$\begin{aligned} I = \int_{-\infty}^{\infty} dg(s) \int_0^s zds' &= - \int_{-\infty}^{0} ds' \int_{-\infty}^{s'} zdg(s) \\ &\quad + \int_0^{\infty} ds' \int_{s'}^{\infty} zdg(s). \end{aligned} \tag{7}$$

As $x^{(n-1)}(s)$ is absolutely continuous,

$$x(s) = x(0) + sx'(0) + \cdots + \frac{s^{n-1}x^{(n-1)}(0)}{(n-1)!} + \int_0^s zds'. \tag{8}$$

Since R vanishes for polynomials of degree $n-1$ and the integral I exists, $R[x]$ exists and

$$\begin{aligned} R[x] &= R\left[\int_0^s zds'\right] \\ &= \int_0^t p(t, s)x^{(n)}(s')ds' - \int_{-\infty}^{\infty} dg(s) \int_0^s zds'. \end{aligned} \tag{9}$$

Furthermore,

$$(10)\qquad \int_0^t p(t, s')x^{(n)}(s')ds' = \int_0^t ds' \int_{-\infty}^{s'} zdg(s) + \int_0^t ds' \int_{s'}^{\infty} zdg(s).$$

This may be proved as follows. By (6),

$$p(t, s') = \int_{-\infty}^{\infty} p(s, s')dg(s) = \int_{-\infty}^{s'} p(s, s')dg(s) + \int_{s'}^{\infty} p(s, s')dg(s).$$

For fixed $t \in \mathfrak{T}$, each of the last two integrals is a measurable, essentially bounded function of s' for s' between 0 and t; hence (10) follows.

By (9), (10) and (7),

$$(11)\qquad R[x] = \int_{-\infty}^{t} ds' \int_{-\infty}^{s'} zdg(s) - \int_{t}^{\infty} ds' \int_{s'}^{\infty} zdg(s) = R^*[x].$$

The last equality follows from (5). Thus $R^*[x]$ and $R[x]$ exist and are equal.

Necessity: Since $R[x]$ and $R^*[x]$ exist and are equal, (11) and (9) hold, and I exists. Furthermore, (11), (9) and (10) imply (7).

This completes the proof of the theorem.

3. **Sufficient conditions.** Put

$$M(s', t) = \begin{cases} \displaystyle\int_{-\infty}^{s'} | p(s, s') | | dg(s) | & \text{if } s' \leqq 0, \\ \displaystyle\int_{s'}^{\infty} | p(s, s') | | dg(s) | & \text{if } s' > 0, \end{cases} \qquad t \in \mathfrak{T}.$$

If the integral

$$J = \int_{-\infty}^{\infty} | x^{(n)}(s') | M(s', t)ds'$$

is finite for all $t \in \mathfrak{T}$, *then* $R[x]$ *and* $R^*[x]$ *exist and are equal, and* $R^*[x]$ *exists as a Lebesgue-Stieltjes integral.*

Proof. The double integral corresponding to I will exist and (7) will hold, by Fubini's theorem, since the right side of (7) is majorized by J. Hence, by the previous theorem, $R[x]$ and $R^*[x]$ exist and are equal.

That $R^*[x]$ exists as a Lebesgue-Stieltjes integral may be seen as follows. Suppose that $t \geqq 0$. ($t < 0$ is treated similarly.) The integrals

$$(12)\qquad \int_{-\infty}^{0} ds' \int_{-\infty}^{s'} zdg(s), \qquad -\int_{t}^{\infty} ds' \int_{s'}^{\infty} zdg(s)$$

are majorized by J. Furthermore, by (6),

$$\int_0^t ds' \int_{-\infty}^{s'} z\,dg(s) = \int_0^t x^{(n)}(s')p(t, s')ds' - \int_0^t ds' \int_{s'}^{\infty} z\,dg(s). \tag{13}$$

Now the last integral in (13) is majorized by J, and the middle integral is on a finite interval. Hence the integrals (12), (13) exist as Lebesgue-Stieltjes integrals. The sum of (13) and the two integrals (12) is precisely $R^*[x]$, by (11).

Note that, by (8), the integral (3) will exist as a Lebesgue-Stieltjes integral, in the present case, if it is true that (3) with $x(s)$ a polynomial of degree $n-1$ exists as a Lebesgue-Stieltjes integral.

Anyone of the following conditions is sufficient to imply the finiteness of J.

(i) For each $t \in \mathfrak{T}$, $M(s', t)$ is absolutely integrable and $x^{(n)}(s')$ is essentially bounded, on $-\infty < s' < \infty$.

(ii) For each $t \in \mathfrak{T}$, $M(s', t)$ is essentially bounded and $x^{(n)}(s')$ is absolutely integrable, on $-\infty < s' < \infty$.

(iii) For each $t \in \mathfrak{T}$, $g(s, t)$ is constant for sufficiently large s and constant for sufficiently small s.

In the particular case in which $R[x]$ is of the form (1), (2),

$$M(s', t) = \begin{cases} \sum_{-\infty < j \leqq s'} p(s', j) \,|\, L(t - j) \,| & \text{if } s' \leqq 0, \\ \sum_{s' \leqq j \leqq \infty} p(j, s') \,|\, L(t - j) \,| & \text{if } s' > 0. \end{cases}$$

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