GENERALIZATION OF AN INEQUALITY OF HEILBRONN AND ROHRBACH

F. A. BEHREND

Let $a_1, \cdots, a_m$ be positive integers and

$$(1)\quad T(a_1, \cdots, a_m) = \begin{cases} 1 - \sum_{\mu_1=1}^{m} \frac{1}{a_{\mu_1}} + \sum_{\mu_1=2}^{m} \sum_{\mu_2=1}^{\mu_1-1} \frac{1}{\{a_{\mu_1}, a_{\mu_2}\}} - \cdots \\ \qquad + \frac{(-1)^m}{\{a_1, \cdots, a_m\}} & \text{for } m > 0, \\ 1 & \text{for } m = 0, \end{cases}$$

where $\{u_1, \cdots, u_r\}$ denotes the least common multiple of $u_1, \cdots, u_r$. H. A. Heilbronn[1] and H. Rohrbach[2] proved that

$$(2)\quad \begin{aligned} T(a_1, \cdots, a_m) &\geqq \left(1 - \frac{1}{a_1}\right) \cdot \cdots \cdot \left(1 - \frac{1}{a_m}\right) \\ &= T(a_1) \cdot \cdots \cdot T(a_m). \end{aligned}$$

The object of this paper is to prove the following generalization of (2):

$$(3)\quad \begin{aligned} T(a_1, \cdots, a_m, b_1, \cdots, b_n) &\geqq T(a_1, \cdots, a_m) T(b_1, \cdots, b_n) \\ &\qquad \text{for } m \geqq 0, n \geqq 0. \end{aligned}$$

$T(a_1, \cdots, a_m)$ may be interpreted as the density of the set S of all positive integers not divisible by any a_μ, that is,

$$T(a_1, \cdots, a_m) = \lim_{z \to \infty} z^{-1} M(z),$$

where $M(z)$ is the number of elements of S not exceeding z.

For the proof of (3) we require the following lemma.

LEMMA. *If $k \geqq 0$, $l \geqq 0$, and $(d, v_\lambda) = 1$ for $\lambda = 1, \cdots, l$, then*

$$T(du_1, \cdots, du_k, v_1, \cdots, v_l) = \frac{1}{d} T(u_1, \cdots, u_k, v_1, \cdots, v_l) + \left(1 - \frac{1}{d}\right) T(v_1, \cdots, v_l).$$

Received by the editors March 14, 1947.

[1] *On an inequality in the elementary theory of numbers*, Proc. Cambridge Philos. Soc. vol. 33 (1937) pp. 207–209.

[2] *Beweis einer zahlentheoretischen Ungleichung*, J. Reine Angew. Math. vol. 177 (1937) pp. 193–196.

PROOF. $T(du_1, \cdots, du_k, v_1, \cdots, v_l)$ is the density of the set S of all x not divisible by any of $du_1, \cdots, du_k, v_1, \cdots, v_l$. We divide S into two sets S_1 and S_2. S_1 contains all elements of S which are divisible by d; they are of the form $x_1 = dy$ subject to the condition that none of $du_1, \cdots, du_k, v_1, \cdots, v_l$ divides dy, which is equivalent to the condition that none of $u_1, \cdots, u_k, v_1, \cdots, v_l$ divides y; the density of the set of integers y is thus $T(u_1, \cdots, u_k, v_1, \cdots, v_l)$ and the density of S_1 becomes $(1/d)T(u_1, \cdots, u_k, v_1, \cdots, v_l)$. S_2 contains all elements of S which are not divisible by d; as $du_1, \cdots, du_k$ do not divide these elements, S_2 consists of all positive integers x_2 not divisible by any of $d, v_1, \cdots, v_l$, and its density is $T(d, v_1, \cdots, v_l)$. Thus we have

$$\begin{aligned} &T(du_1, \cdots, du_k, v_1, \cdots, v_l) \\ &\qquad = \frac{1}{d} T(u_1, \cdots, u_k, v_1, \cdots, v_l) + T(d, v_1, \cdots, v_l). \end{aligned} \tag{4}$$

Note that this proof still holds when one or both of $k, l = 0$; for $k = 0$, (4) reduces to

$$T(v_1, \cdots, v_l) = \frac{1}{d} T(v_1, \cdots, v_l) + T(d, v_1, \cdots, v_l), \tag{5}$$

whence

$$T(d, v_1, \cdots, v_l) = \left(1 - \frac{1}{d}\right) T(v_1, \cdots, v_l). \tag{6}$$

Substituting (6) in (4) we obtain the lemma.

PROOF OF (3): by induction with respect to $N = a_1 + \cdots + a_m + b_1 + \cdots + b_n$. For $N = 0$, $m = n = 0$ and the three T's in (3) reduce to 1. Assume that (3) holds for $N' < N$.

First case: Any two of $a_1, \cdots, a_m, b_1, \cdots, b_l$ relatively prime. In this case

$$\begin{aligned} &T(a_1, \cdots, a_m, b_1, \cdots, b_n) \\ &\quad = \left(1 - \frac{1}{a_1}\right) \cdot \cdots \cdot \left(1 - \frac{1}{a_m}\right) \cdot \left(1 - \frac{1}{b_1}\right) \cdot \cdots \cdot \left(1 - \frac{1}{b_n}\right) \\ &\quad = T(a_1, \cdots, a_m) T(b_1, \cdots, b_n). \end{aligned}$$

Second case: s exists such that $2 \leqq s \leqq m + n$ and (i) certain s of the $a_1, \cdots, a_m, b_1, \cdots, b_n$ have a common divisor $d > 1$, (ii) any $s + 1$ of them have the greatest common divisor 1 (this condition being vacuous for $s=m+n$). Rearranging the $a_1, \cdots, a_m, b_1, \cdots, b_n$ we may assume that $a_1, \cdots, a_\mu, b_1, \cdots, b_\nu$ have the common divisor $d>0$ where $\mu+\nu=s$ (μ or ν may be 0); then

$$\begin{aligned} a_\rho &= d\bar{a}_\rho \quad \text{for} \quad \rho \leqq \mu; \qquad (a_\rho, d) = 1 \quad \text{for} \quad \rho > \mu; \\ b_\sigma &= d\bar{b}_\sigma \quad \text{for} \quad \sigma \leqq \nu; \qquad (b_\sigma, d) = 1 \quad \text{for} \quad \sigma > \nu. \end{aligned}$$

By the lemma

$$\begin{aligned} &T(a_1, \cdots, a_m)T(b_1, \cdots, b_n) \\ &= T(d\bar{a}_1, \cdots, d\bar{a}_\mu, a_{\mu+1}, \cdots, a_m)T(d\bar{b}_1, \cdots, d\bar{b}_\nu, b_{\nu+1}, \cdots, b_n) \\ &= \left(\frac{1}{d} T(\bar{a}_1, \cdots, \bar{a}_\mu, a_{\mu+1}, \cdots, a_m) + \left(1-\frac{1}{d}\right) T(a_{\mu+1}, \cdots, a_m)\right) \\ &\quad \cdot \left(\frac{1}{d} T(\bar{b}_1, \cdots, \bar{b}_\nu, b_{\nu+1}, \cdots, b_n) + \left(1-\frac{1}{d}\right) T(b_{\nu+1}, \cdots, b_n)\right) \\ &= \frac{1}{d} T(\bar{a}_1, \cdots, \bar{a}_\mu, a_{\mu+1}, \cdots, a_m) T(\bar{b}_1, \cdots, \bar{b}_\nu, b_{\nu+1}, \cdots, b_n) \\ &\quad + \left(1-\frac{1}{d}\right) T(a_{\mu+1}, \cdots, a_m) T(b_{\nu+1}, \cdots, b_n) \\ &\quad - \frac{1}{d}\left(1-\frac{1}{d}\right)\left(T(a_{\mu+1}, \cdots, a_m) - T(\bar{a}_1, \cdots, \bar{a}_\mu, a_{\mu+1}, \cdots, a_m)\right) \\ &\quad \cdot \left(T(b_{\nu+1}, \cdots, b_n) - T(\bar{b}_1, \cdots, \bar{b}_\nu, b_{\nu+1}, \cdots, b_n)\right). \end{aligned} \tag{7}$$

Observing that the first two terms may be estimated by the induction hypothesis and that the factors of the third term are not less than 0, we get

$$\begin{aligned} &T(a_1, \cdots, a_m)T(b_1, \cdots, b_n) \\ &\qquad \leqq \frac{1}{d} T(\bar{a}_1, \cdots, \bar{a}_\mu, a_{\mu+1}, \cdots, a_m, \bar{b}_1, \cdots, \bar{b}_\nu, b_{\nu+1}, \cdots, b_n) \\ &\qquad\quad + \left(1 - \frac{1}{d}\right) T(a_{\mu+1}, \cdots, a_m, b_{\nu+1}, \cdots, b_n) \\ &\qquad = T(a_1, \cdots, a_m, b_1, \cdots, b_n) \end{aligned} \tag{8}$$

by the lemma. Hence (3) is proved.

It is easy to decide when equality holds in (3). Equality will certainly hold if $(a_\rho, b_\sigma)=1$ for $\rho=1, \cdots, m$, $\sigma=1, \cdots, n$; this can be seen on the lines of the above proof, or, directly, by substituting the explicit value (1) of T into (3) and observing that $\{a_{\rho_1}, \cdots, a_{\rho_p}\} \cdot \{b_{\sigma_1}, \cdots, b_{\sigma_q}\} = \{a_{\rho_1}, \cdots, a_{\rho_p}, b_{\sigma_1}, \cdots, b_{\sigma_q}\}$. The converse is obviously not true as, for example, $T(2, 4)T(3, 6)=T(2, 4, 3, 6)$; the reason is that, in this example, the numbers 4, 6 are redundant; the example may be written simpler $T(2)T(3)=T(2, 3)$. In general u_k will be redundant in $T(u_1, \cdots, u_k)$ if it is a multiple of another u_λ. If redundant elements in $T(a_1, \cdots, a_m)$ and $T(b_1, \cdots, b_n)$ are removed, the converse of the above statement can be proved: If for some ρ, σ $(a_\rho, b_\sigma)>1$, inequality holds in (3). We may assume that $(a_1, b_1)>1$ and can apply (7) with $\mu \geqq 1$, $\nu \geqq 1$. Now, if u is not divisible by any of $v_1, \cdots, v_l$, then

$$T(v_1, \cdots, v_l) > T(u, v_1, \cdots, v_l),$$

for the set of positive integers not divisible by $v_1, \cdots, v_l$ contains the numbers $u(v_1 \cdot \cdots \cdot v_l z+1)$, $z=0, 1, 2, \cdots$, which possess a positive density and are not contained in the set of numbers not divisible by $u, v_1, \cdots, v_l$. As a_1, and hence $\bar{a}_1$, is not a multiple of any of $a_{\mu+1}, \cdots, a_m$ and $\bar{b}_1$ not a multiple of any of $b_{\nu+1}, \cdots, b_n$, it follows that the factors of the last term of (7) are positive, and the inequality sign will hold in (8).

THE UNIVERSITY OF MELBOURNE