ON THE FACTORISATION OF ORTHOGONAL TRANSFORMATIONS INTO SYMMETRIES

H. C. LEE

Consider the quadratic form

$$g_{11}(x^1)^2 + g_{22}(x^2)^2 + \cdots + g_{nn}(x^n)^2$$

where each g_{ii} is $+1$ or -1 (but fixed once for all). A homogeneous linear transformation in the variables $x^1, \cdots, x^n$ (or its matrix) is called *orthogonal* if it leaves this form invariant. If and only if all g_{ii} have the same sign we have orthogonality in the classical sense. In any sense it can be easily seen that the orthogonal transformations form a group and that they have determinants ± 1.

All quantities considered may be either in the real field or in the complex field (or in any field whatever).

Let $g_{ij}=0$ $(i\neq j)$. Then, if s^i is a contravariant vector, its covariant is defined by $s_i = g_{ik}s^k$ (summation convention). The vector s is said to be *isotropic* if it has zero length, that is, if $s^h s_h = 0$. If s is not an isotropic vector, the matrix

$$\overset{i}{s}_k = \delta^i_k - 2\overset{i}{s}s_k/\overset{h}{s}s_h,$$

where $\delta^i_k = 1$ or 0 if $i=k$ or $i\neq k$, defines a special orthogonal transformation of determinant -1 which we call a *symmetry*.[1]

Now, by a series of arguments, E. Cartan has proved the following theorem.

THEOREM.[2] *Every orthogonal transformation is decomposable into the product of a number not greater than n of symmetries.*

I shall give a short proof[3] of this theorem below:

Let a^i_k be the matrix of an arbitrarily given orthogonal transformation. Form the product matrix

$$\overset{i}{b}_k = \overset{i}{s}_j\overset{j}{a}_k = \overset{i}{a}_k - 2\overset{i}{s}s_j\overset{j}{a}_k/\overset{h}{s}s_h$$

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[1] E. Cartan, *La théorie des spineurs*, vol. 1, p. 13.

[2] Cartan, loc. cit. pp. 13–17.

[3] The first half of the present proof is similar to a part of another simple proof given by W. Givens, *Factorization and signatures of Lorentz matrices*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 82, but Givens' proof only establishes the weaker statement that the number of symmetries in the factorisation is not greater than $2n$.

and consider its first column b^i_1. We show that one can choose the vector s such that $b^i_1 = \delta^i_1$. In fact, it suffices for this to take $s^i = \rho(a^i_1 - \delta^i_1)$, $\rho \neq 0$. Then we find

$$s^h s_h = g_{hi} s^h s^i = 2\rho^2 (g_{11} - g_{1i} a^i_1) = 2\rho^2 g_{11}(1 - a^1_1).$$

(1) If $a^1_1 \neq 1$, we have $s^h s_h \neq 0$ and so s is not an isotropic vector. Then

$$b^i_1 = a^i_1 - 2s^i s_j a^j_1 / 2\rho^2 g_{11}(1 - a^1_1) = \delta^i_1.$$

This being the first column of the matrix b^i_k, the orthogonality between this column and each of the other columns implies that the matrix b has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$$

where c is evidently an orthogonal matrix of order $n-1$. Hence, by induction, the theorem is proved in this case.

(2) If $a^1_1 = 1$, we can evidently transform the given orthogonal transformation by another orthogonal transformation so as to render $a^1_1 \neq 1$, and we have only to observe that the symmetries are transformed again into symmetries by an orthogonal transformation.

CAMBRIDGE, ENGLAND