A REFINEMENT OF PELLET'S THEOREM

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1. **Introduction.** S. Lipka[1] has recently announced a refinement of the classic theorem of Cauchy that all the zeros of the polynomial

$$(1.1) \qquad f(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad a_n \neq 0,$$

lie in the circle $|z| \leqq r$, where r is the positive root of the real equation

$$(1.2) \qquad F_n(z) = |a_0| + |a_1| z + \cdots + |a_{n-1}| z^{n-1} - |a_n| z^n = 0.$$

Lipka's refinement consists in replacing the circle $|z| = r$ by a curve $G(r_0, r; n, \alpha_0)$ which bounds a gear-wheel region. This region is formed by deleting from the circle $|z| \leqq r$ the points common to the annular ring $0 < r_0 < |z| \leqq r$ and to the n sectors

$$(1.3) \qquad \frac{\alpha_0}{n} - \frac{\pi}{2n} + \frac{2\pi_k}{n} \leqq \arg z \leqq \frac{\alpha_0}{n} + \frac{\pi}{2n} + \frac{2\pi k}{n},$$

$k = 0, 1, \cdots, n-1$. In these formulas r_0 is the positive root of the equation

$$(1.4) \qquad \Phi_n(z) = |a_1| + |a_2| z + \cdots + |a_{n-1}| z^{n-2} - |a_n| z^{n-1} = 0$$

and $\alpha_0 = \arg a_0/a_n$.

Now, the Cauchy theorem is but a special case of the following theorem due to Pellet.[2]

PELLET'S THEOREM. *If the polynomial*

$$(1.5) \qquad f(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots + a_n z^n, \qquad a_p \neq 0,$$

is such that the real polynomial

$$(1.6) \qquad \begin{aligned} F_p(z) = {} & |a_0| + |a_1| z + \cdots + |a_{p-1}| z^{p-1} - |a_p| z^p \\ & + |a_{p+1}| z^{p+1} + \cdots + |a_n| z^n \end{aligned}$$

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[1] S. Lipka, Monatshefte für Mathematik und Physik vol. 50 (1944) pp. 209–221.

[2] A. Pellet, Bull. Sci. Math. vol. 5 (1881) pp. 393–395. The converse to this theorem was discussed by J. L. Walsh, Ann. of Math. vol. 26 (1924–1925) pp. 59–64 and A. Ostrowski, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 742–746. See also M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, chap. 7, to be published as a volume of Mathematical Surveys.

has two positive zeros r and R with $r<R$, *then* $f(z)$ *has exactly* p *zeros in or on the circle* $|z|\leqq r$ *and no zeros in the annular ring* $r<|z|<R$.

It is Pellet's theorem which we propose to refine as indicated in the following theorem.

THEOREM 1.1. *Under the hypotheses of Pellet's Theorem the polynomial*

$$\begin{aligned}\Phi_p(z) = {} & |a_1| + |a_2|z + \cdots + |a_{p-1}|z^{p-2} - |a_p|z^{p-1} \\ & + |a_{p+1}|z^p + \cdots + |a_n|z^{n-1}\end{aligned} \tag{1.7}$$

has also two positive zeros r_0 *and* R_0 *with*

$$r_0 < r < R < R_0. \tag{1.8}$$

Furthermore, $f(z)$ *has exactly* p *zeros in or on the curve* $G(r_0, r; p, \alpha_0)$ *where* $\alpha_0 = \arg a_0/a_p$ *and no zeros between the curves* $G(r_0, r; p, \alpha_0)$ *and* $G(R, R_0; p, \alpha_0+\pi)$.

Theorem 1.1 will be proved in §2 and applied in §3 to the refinement of various known bounds on the zeros of a polynomial. Finally, the theorem will be generalized in §4, first by replacing the polynomial $\Phi_p(z)$ by the polynomial $\Phi_{kp}(z) = F_p(z) - |a_k|z^k$ and secondly by replacing the polynomial $f(z)$ by a power series.

2. **Proof of Theorem 1.1.** Let us first prove the existence of the roots r_0 and R_0 of equation $\Phi_p(z)=0$ and the validity of inequality (1.8). Since r and R are the positive zeros of $F_p(z)$, it follows from (1.6) that, for any sufficiently small positive number ϵ,

$$F_p(\rho) < 0 \qquad \text{if } r+\epsilon \leqq \rho \leqq R-\epsilon. \tag{2.1}$$

In view of the equation

$$F_p(z) = |a_0| + z\Phi_p(z), \tag{2.2}$$

the zeros r and R of $F_p(z)$ satisfy the relations

$$\Phi_p(r) = -|a_0|/r < 0, \qquad \Phi_p(R) = -|a_0|/R < 0. \tag{2.3}$$

When taken together with the facts that

$$\Phi_p(0) > 0, \qquad \Phi_p(\infty) > 0, \tag{2.4}$$

the relations (2.3) imply the existence of two positive zeros r_0 and R_0 of $\Phi_p(z)$ and the validity of inequality (1.8), as well as the inequality

$$\Phi_p(\rho) < 0 \tag{2.5}$$

for $r_0+\epsilon \leqq \rho \leqq R_0-\epsilon$.

Let us now set $z=\rho e^{i\theta}$ and

$$a_k/a_p = A_k e^{\alpha_k i}, \qquad k = 0, 1, \cdots, n. \tag{2.6}$$

In this notation, the real part of $\rho^p f(z)/a_p z^p$ is

$$\begin{aligned} \text{Re}\, [\rho^p f(z)/a_p z^p] &= \sum_{j=0}^{p-1} A_j \rho^j \cos\, [(p-j)\theta - \alpha_j] + \rho^p \\ &\quad + \sum_{j=p+1}^{n} A_j \rho^j \cos\, [(j-p)\theta + \alpha_j] \end{aligned} \tag{2.}$$

and the inequalities (2.1) and (2.5) become

$$\rho^p > A_0 + A_1\rho + \cdots + A_{p-1}\rho^{p-1} + A_{p+1}\rho^{p+1} + \cdots + A_n\rho^n \tag{2.8}$$

for $r+\epsilon \leqq \rho \leqq R-\epsilon$, and

$$\rho^p > A_1\rho + A_2\rho^2 + \cdots + A_{p-1}\rho^{p-1} + A_{p+1}\rho^{p+1} + \cdots + A_n\rho^n \tag{2.9}$$

for $r_0+\epsilon \leqq \rho \leqq R_0-\epsilon$.

On substituting from inequality (2.8) into (2.7), we find

$$\begin{aligned} \text{Re}\, (\rho^p f(z)/a_p z^p) &> \sum_{j=0}^{p-1} A_j \rho^j \{\cos\, [(p-j)\theta - \alpha_j] + 1\} \\ &\quad + \sum_{j=p-1}^{n} A_j \rho^j \{\cos\, [(j-p)\theta + \alpha_j] + 1\} \geqq 0 \end{aligned} \tag{2.10}$$

for $r+\epsilon \leqq \rho \leqq R-\epsilon$. On substituting from inequality (2.9) into (2.7), we find

$$\begin{aligned} \text{Re}\, (\rho^p f(z)/a_p z^p) &> A_0 \cos\, (p\theta - \alpha_0) \\ &\quad + \sum_{j=1}^{p-1} A_j \rho^j \{\cos\, [(p-j)\theta - \alpha_j] + 1\} \\ &\quad + \sum_{j=p-1}^{n} A_j \rho^j \{\cos\, [(j-p)\theta + \alpha_j] + 1\} \end{aligned} \tag{2.11}$$

for $r_0+\epsilon \leqq \rho \leqq R_0-\epsilon$. The right side of (2.11) is surely non-negative if θ is such that $\cos\, (p\theta - \alpha_0) \geqq 0$, that is, such that

$$-\frac{\pi}{2} + 2\pi k \leqq p\theta - \alpha_0 \leqq \frac{\pi}{2} + 2\pi k,$$

where k is an integer; that is, if

$$\frac{\alpha_0}{p} - \frac{\pi}{2p} + \frac{2\pi k}{p} \leqq \theta \leqq \frac{\alpha_0}{p} + \frac{\pi}{2p} + \frac{2\pi k}{p}, \qquad k = 0, 1, \cdots, p.$$

In other words,

$$\text{Re}\,(\rho^p f(z)/a_p z^p) > 0 \tag{2.12}$$

and hence $f(z) \neq 0$ at all points z between the curves $G(r_0, r; p, \alpha_0)$ and $G(R, R_0; p, \alpha_0+\pi)$.

Inequality (2.12) also may be used to show that in or on the curve $G(r_0, r; p, \alpha_0)$, there are exactly p zeros of $f(z)$. For, let us consider the net change $\Delta_{G_\epsilon} \arg w$ in the argument of the point $w = [\rho^p f(z)/a_p z^p]$ as z describes counterclockwise the curve $G_\epsilon = G(r_0+\epsilon, r+\epsilon; p, \alpha_0)$ where ϵ is a small positive number. Since Re $(w)>0$, w describes a closed curve entirely in the right-half w-plane. That is, $\Delta_{G_\epsilon} \arg w = 0$ on this curve. This means that the function w has as many zeros as poles in the curve G_ϵ and this, in turn, means that $f(z)$ has precisely p zeros in G_ϵ for every sufficiently small positive ϵ.

3. **Applications.** Let us first apply Theorem 1.1 to the class of polynomials

$$\begin{aligned} f(z) &= b_0 e^{i\beta_0} + (b_1 - b_0)e^{i\beta_1}z + \cdots \\ &\quad + (b_m - b_{m-1})e^{i\beta_m}z^m - b_m e^{i\beta_{m+1}}z^{m+1} \end{aligned} \tag{3.1}$$

where the b_j are real numbers such that

$$b_{p-1} < b_{p-2} < \cdots < b_0 < 0 < b_m < b_{m-1} < \cdots < b_p. \tag{3.2}$$

The corresponding polynomials $F_p(z)$ and $\Phi_p(z)$ are

$$\begin{aligned} F_p(z) &= -b_0 + (b_0 - b_1)z + \cdots + (b_{p-2} - b_{p-1})z^{p-1} \\ &\quad - (b_p - b_{p-1})z^p + (b_p - b_{p+1})z^{p+1} + \cdots \\ &\quad + (b_{m-1} - b_m)z^m + b_m z^{m+1}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \Phi_p(z) &= (b_0 - b_1) + \cdots + (b_{p-2} - b_{p-1})z^{p-2} \\ &\quad - (b_p - b_{p-1})z^{p-1} + (b_p - b_{p+1})z^p + \cdots \\ &\quad + (b_{m-1} - b_m)z^{m-1} + b_m z^m. \end{aligned} \tag{3.4}$$

On defining

$$g(z) = b_0 + b_1 z + \cdots + b_m z^m, \tag{3.5}$$

we may write

$$F_p(z) = (z-1)g(z), \qquad z\Phi_p(z) = b_0 + g(z)(z-1).$$

Clearly $F_p(1)=0$. Since $F_p(1+\delta) = \delta g(1+\delta)$, then for δ sufficiently small $g(1)>0$ implies that $F_p(1+\delta)>0$ or <0 according as $\delta>0$ or <0 and $g(1)<0$ implies that $F_p(1+\delta)<0$ or >0 according as $\delta>0$ or

<0. That is, using the notation of Theorem 1.1, we see that

$$r_0 < r < 1 = R < R_0 \qquad \text{if } g(1) > 0,$$

$$r_0 < r = 1 < R < R_0 \qquad \text{if } g(1) < 0,$$

$$\alpha_0 = \beta_0 - \beta_p + \pi.$$

We thereby conclude that the following is true.

THEOREM 3.1. *Let $f(z)$, $\Phi_p(z)$ and $g(z)$ denote the polynomials* (3.1), (3.4) *and* (3.5) *respectively. Then, if $g(1)>0$, $f(z)$ has exactly p zeros in the curve $G(r_0, 1; p, \beta_0-\beta_p+\pi)$ and $g(z)$ has p zeros in the curve $G(r_0, 1; p, \pi)$. If $g(1)<0$, $f(z)$ has p zeros in or on the curve $G(r_0, 1; p, \beta_0-\beta_p+\pi)$ and $g(z)$ has $p-1$ zeros in or on the curve $G(r_0, 1; p, \pi)$.*

An analogous result for $g(z)$ with, however, curve $G(r_0, 1; p, \pi)$ replaced by the circle $|z|=1$ was first stated by Berwald.[3] His result was a generalization of the Kakeya-Eneström[4] theorem that all the zeros of the real polynomial (3.5) with $0<b_0<b_1<\cdots<b_n$ lie in or on the unit circle $|z|=1$. Our analogy to the Kakeya-Eneström theorem will be included in the following theorem.

THEOREM 3.2. *Every polynomial of the form*

$$f(z) = \sum_{j=0}^{n} (b_j - b_{j-1})e^{i\beta_j}z^j, \qquad b_{-1} = b_n = 0 < b_0 < b_1 < \cdots < b_{n-1},$$

has all of its zeros in or on the curve $G(r_0, 1; n, \beta_0-\beta_n+\pi)$ where r_0 is the positive root of the equation

$$\Phi_n = (b_1 - b_0) + (b_2 - b_1)z + \cdots + (b_{n-1} - b_{n-2})z^{n-2} - b_{n-1}z^{n-1} = 0.$$

Furthermore, every polynomial of the form

$$g(z) = b_0 + b_1z + \cdots + b_{n-1}z^{n-1}, \qquad 0 < b_0 < b_1 < \cdots < b_{n-1},$$

has all of its zeros in or on the curve $G(r_0, 1; n, \pi)$.

This theorem may be derived from Theorem 3.1 indirectly by a limiting process or directly by the same methods as used for Theorem 3.1.

In our next application, we shall use Theorem 1.1 just in the case $p=n$. This restriction is made only to simplify the statement of results, since a similar application may be made when p is an arbitrary integer, $0<p\leqq n$. The result to be proved is the following.

[3] L. Berwald, Math. Zeit. vol. 37 (1933) pp. 61–76.

[4] S. Kakeya, Tôhoku Math. J. vol. 2 (1912) pp. 140–142 and G. Eneström, Ibid. vol. 18 (1920) pp. 34–36.

THEOREM 3.3. *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\mu_1, \mu_2, \cdots, \mu_{n-1}$ be any two sets of numbers such that*

$$\sum_{j=1}^{n} (1/\lambda_j) = 1, \quad \sum_{j=1}^{n-1} (1/\mu_j) = 1; \qquad 0 < \mu_j \leqq \lambda_j,\ j = 1, 2, \cdots, n-1.$$

For the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, let

$$M = \max\ [\lambda_k |a_{n-k}| / |a_n|]^{1/k}, \qquad k = 1, 2, \cdots, n, \tag{3.6}$$

$$M_0 = \max\ [\mu_k |a_{n-k}| / |a_n|]^{1/k}, \qquad k = 1, 2, \cdots, n-1. \tag{3.7}$$

Then all the zeros of $f(z)$ lie in or on the curve $G(M_0, M; n, \alpha_0)$, where $\alpha_0 = \arg(a_0/a_n)$.

From (3.6) and (3.7), obviously $0 < M_0 < M$. Also,

$$\lambda_k |a_{n-k}| \leqq |a_n| M^k, \qquad \mu_k |a_{n-k}| \leqq |a_n| M_0^k$$

and thus

$$\sum_{k=1}^{n} |a_{n-k}| M^{n-k} \leqq \sum_{k=1}^{n} (1/\lambda_k) |a_n| M^n = |a_n| M^n, \tag{3.8}$$

$$\sum_{k=1}^{n-1} |a_{n-k}| M_0^{n-k} \leqq \sum_{k=1}^{n-1} (1/\mu_k) |a_n| M_0^n = |a_n| M_0^n. \tag{3.9}$$

An equality in (3.8) would imply that M is the positive root r of the equation (1.2) whereas an inequality in (3.8) would imply that $M > r$. Likewise, an equality in (3.9) would imply that M_0 is the positive root r_0 of the equation (1.4) whereas an inequality in (3.9) would imply that $M_0 > r_0$. Since by Theorem 1.1 all the zeros of $f(z)$ lie in or on the curve $G(r_0, r; n, \alpha_0)$, they surely all lie in or on the curve $G(M_0, M; n, \alpha_0)$.

Theorem 3.3 whose proof we have just completed is a refinement of the result due to Fujiwara[5] that all the zeros of $f(z)$ lie in or on the circle $|z| \leqq M$.

As a simple application of Theorem 3.3, let us take $\lambda_j = n$ for $j = 1, 2, \cdots, n$ and $\mu_j = n-1$ for $j = 1, 2, \cdots, n-1$. We obtain thereby the following corollary.

COROLLARY 3.3a. *For the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ let $N = \max\ [n|a_{n-k}/a_n|]^{1/k}$, $k = 1, 2, \cdots, n$, and $N_0 = \max\ [(n-1)|a_{n-k}/a_n|]^{1/k}$, $k = 1, 2, \cdots, n-1$. Then all the zeros of $f(z)$ lie in or on the curve $G(N_0, N; n, \alpha_0)$ where $\alpha_0 = \arg(a_0/a_n)$.*

[5] M. Fujiwara, Tôhoku Math. J. vol. 10 (1916) pp. 167–171.

As another simple application of Theorem 3.3, let us take

$$\lambda_k = \sum_{j=0}^{n-1} |a_j| / |a_{n-k}|, \qquad k = 0, 1, 2, 3, \cdots, n,$$

$$\mu_k = \sum_{j=1}^{n-1} |a_j| / |a_{n-k}|, \qquad k = 0, 1, 2, \cdots, n-1.$$

Clearly,

$$\sum_{j=1}^{n} 1/\lambda_k = 1, \qquad \sum_{j=1}^{n-1} 1/\mu_j = 1.$$

Here

$$M = \max \left[\sum_{j=0}^{n-1} |a_j| / |a_n| \right]^{1/k} = \lambda_0 \quad \text{or} \quad \lambda_0^{1/n}$$

according as $\lambda_0 > 1$ or < 1, and

$$M_0 = \max \left[\sum_{j=1}^{n-1} |a_j| / |a_n| \right]^{1/k} = \lambda_0 \quad \text{or} \quad \mu_0^{1/n}$$

according as $\mu_0 > 1$ or < 1. We thereby obtain the following corollary.

COROLLARY 3.3b. *For the polynomial* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, *let*

$$\lambda_0 = \sum_{j=0}^{n-1} |a_j| / |a_n| \quad \text{and} \quad \mu_0 = \sum_{j=1}^{n-1} |a_j| / |a_n|.$$

Let $\gamma = \lambda_0$ *or* $\lambda_0^{1/n}$ *according as* $\lambda_0 > 1$ *or* < 1, *and let* $\delta = \mu_0$ *or* $\mu_0^{1/n}$ *according as* $\mu_0 > 1$ or < 1. *Then all the zeros of* $f(z)$ *lie in or on the curve* $G(\delta, \gamma; n, \alpha_0)$ *where* $\alpha_0 = \arg a_0/a_n$.

4. **Generalizations.** Let us define $\Psi_{kp}(z) = F_p(z) - |a_k| z^k$, $k \neq p$. Since $\Psi_{0p}(z) = z\Phi_p(z)$, the positive zeros of $\Phi_p(z)$ are also the positive zeros of $\Psi_{0p}(z)$. By modifying somewhat the details of proof of Theorem 1.1, we may prove the following generalization.

THEOREM 4.1. *Under the hypotheses of Pellet's Theorem the polynomial*

$$\Psi_{kp}(z) = F_p(z) - |a_k| z^k, \qquad k \neq p, \ a_k \neq 0,$$

has also two positive zeros r_k *and* R_k *with* $r_k < r < R < R_k$. *Furthermore* $f(z)$ *has exactly* p *zeros in or on the curve* $G(r_k, r; p-k, \alpha_k)$ *where* $\alpha_k = \arg (a_k/a_p)$ *and none between the curves* $G(r_k, r; p-k, \alpha_k)$ *and* $G(R, R_k; p-k, \alpha_k + \pi)$.

Our final generalization will consist in replacing the polynomial $f(z)$ of Theorem 4.1 by a power series.

THEOREM 4.2. *If the power series*

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots, \qquad a_k a_p \neq 0,$$

having a radius of convergence of $\rho, 0 < \rho \leqq \infty$, *is such that each polynomial*

$$F_{np}(z) = |a_0| + |a_1| z + \cdots + |a_{p-1}| z^{p-1} - |a_p| z^p + |a_{p+1}| z^{p+1} + \cdots + |a_n| z^n$$

with $n \geqq N > p$ *has a positive zero* $r^{(n)}$, $r^{(n)} \leqq \rho_1 < \rho$, *then the function* $F_p(z) = \lim_{n=\infty} F_{np}(z)$ *has a positive zero* $r < \rho$; *the function*

$$\Psi_{kp}(z) = F_p(z) - |a_k| z^k, \qquad k \neq p,$$

has a positive zero r_k, $r_k < r < \rho$, *and the function* $f(z)$ *has exactly* p *zeros in or on the curve* $G(r_k, r; p-k, \alpha_k)$ *and hence in the curve* $G(r_k, \rho; p-k, \alpha_k)$.

This theorem results from Theorem 4.1 on the use of the Hurwitz theorem that within its circle of convergence a non-constant power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$ has as zeros the limit points of the zeros of the polynomials $f_n(z) = \sum_{j=0}^{n} a_j z^j$.

If $F_{np}(z)$ has two positive zeros in $|z| < \rho$, we may choose $r^{(n)}$ as the smaller one. Letting

$$\Psi_{nkp}(z) = F_{np}(z) - |a_k| z^k,$$

we see that $\Psi_{nkp}(z)$ has a positive zero $r_k^{(n)}$, $r_k^{(n)} < r^{(n)}$. Clearly, the power series $F_p(z)$ and $\Psi_{kp}(z)$ have the same radius ρ of convergence and have respectively the positive zeros $r = \lim_{n=\infty} r^{(n)}$ and $r_k = \lim_{n=\infty} r_k^{(n)}$, with $r_k < r < \rho$. Now, given any small positive ϵ, we can find an $N > 0$ such that the circle of radius ϵ drawn about the point $z = r$ will contain $r^{(n)}$ for all $n \geqq N$ and the circle of radius ϵ drawn about $z = r_k$ will contain $r_k^{(n)}$ for all $n \geqq N$. This means that in or on the curve $G(r_k + \epsilon, r + \epsilon; p-k, \alpha_k)$, which for any sufficiently small positive ϵ is contained in the circle $|z| < \rho$, lie exactly p zeros of each polynomial $f_n(z)$ for all $n \geqq N$. Since a circle of radius ϵ about any zero of $f(z)$ in $|z| < \rho$ contains a zero of each $f_n(z)$, $n \geqq N$, it follows that in or on the curve $G(r_k + \epsilon, r + \epsilon; p-k, \alpha_k)$ lie exactly p zeros of $f(z)$. Since ϵ is an arbitrary, small positive number, it follows that exactly p zeros of $f(z)$ lie in or on the curve $G(r_k, r; p-k, \alpha_p)$ as stated in Theorem (4.2).

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