ON UNIQUE INVARIANT MEASURES

HENRY M. SCHAERF

1. Statement of the problems. Let S be a σ-field[1] of subsets (measurable sets) of an abstract group G. What can be said about the structure of S if there is a unique measure defined on S and invariant under the group operation? What are the conditions for the uniqueness of an invariant measure? These are the problems studied in this note by means of a simple lemma.

2. Definitions and results.

DEFINITION 1. "*Measure*" means in this paper a non-negative, countably additive function of the set $X \in S$ such that G is not of measure 0 and is the union of a sequence of measurable sets of finite measure. For any two measures m and n we denote by $S_{m,n}$ the σ-field of subsets of $G \times G$, defined so as to allow the application of the generalized theorem of Fubini [1, p. 87].[2]

DEFINITION 2. A measure m is called *invariant* if $A \in S$, $g \in G$ implies $gA \in S$ and $m(gA) = m(A)$. An invariant measure is called *unique* if it differs from any other invariant measure only by a multiplicative constant.

FUNDAMENTAL ASSUMPTION. It is assumed that $g \in G$, $A \in S$ implies $Ag \in S$ and that any two invariant measures m, n satisfy the following *condition* M_1: The transformation $[(x, y) \to (y^{-1}x, y)]$ sends every set $A \times G$ with $A \in S$ into a set of $S_{m,n}$.

DEFINITION 3. A measurable set A is called *almost congruent by finite* (*resp. denumerable*) *partition* with the measurable set A' if there is a finite (resp. infinite) sequence of disjoint measurable subsets A_k of A with $m(A - \bigcup_k A_k) = 0$ and a corresponding sequence of elements g_k of G such that the sets $g_k^{-1} A_k$ are disjoint subsets of A' and $m(A' - \bigcup_k g_k^{-1} A_k) = 0$.

The answer to our first problem is given by the following theorem.

THEOREM 1. *If the measure is unique invariant then any measurable set A, whose measure is not greater than that of a measurable set B or equal to it, is almost congruent by finite or denumerable partition with some measurable subset of B or with B, respectively.*

Presented to the Society, November 29, 1947; received by the editors July 25, 1947, and, in revised form, August 18, 1947.

[1] That is, a class of sets containing G and closed under complementation and the formation of countable unions.

[2] Numbers in brackets refer to the bibliography at the end.

COROLLARY. *A unique invariant measure m accepts any of its values that are less than m(B) on some subset of B.*[3]

The answer to our second problem is contained in the following theorem.

THEOREM 2. *In order that an invariant measure m be unique, it is necessary and sufficient that $m(Xg)$ be an absolutely continuous function of the measurable set X for every g in G.*

COROLLARY. *A bi-invariant measure is unique.*

To formulate two easy consequences of Theorem 2 we introduce the following definition.

DEFINITION 4. The measure m satisfies the *condition* M_2 (resp. M_2^*) if the transformation $[(x, y) \to (yx, y)]$ (resp. $[(x, y) \to (xy, y)]$) sends every set $A \times G$ with $A \in S$ into a set of $S_{m,m}$.

THEOREM 3. *Every invariant measure which satisfies the condition* M^2 *is unique.*

THEOREM 3*. *Every invariant measure which satisfies the condition* M_2^* *is unique.*

Theorem 3 has been found before by A. Weil [2, pp. 140–149]. Its new proof given here seems to be simpler and more natural than that of Weil. Theorem 3* is new and completes Weil's result.

3. **Two lemmas.** The proof of the above theorems relies upon 2 lemmas.

LEMMA 1. *Let n and m be measures satisfying the condition* M_1, *and A and B measurable sets. Then we have*

$$(3.1)\qquad \int_B m(Ax^{-1})dn(x) = \int_G n(y^{-1}A \cap B)dm(y),$$

$$(3.2)\qquad \int_B m(x^{-1}A)dn(x) = \int_G n(Ay^{-1} \cap B)dm(y).$$

Indeed, let H be the image of the set $A \times G$ by the transformation $[(x, y) \to (y^{-1}x, y)]$ and $f_1(x, y)$ resp. $f_2(x, y)$ the characteristic functions of $H \cap B \times G$ resp. $H \cap G \times B$.

Then the above relations result from the evaluation of the integrals

[3] It follows from [4] that these values either are multiples of a positive number or fill a closed interval (which can be infinite).

of $f_1(x, y)$ resp. $f_2(x, y)$ over $G\times G$ by the generalized theorem of Fubini [2, p. 87].

Remark. The relations (3.1), (3.2) remain valid for the group operation $x*y=y\cdot x$.[4] So do the theorems 1 and 2, as they will be deduced from Lemma 1.

LEMMA 2. *Let m be a measure such that $m(Xg)$ is an absolutely continuous function of the measurable set X for every g in G. Then:*

(a) *For any two measurable sets A, B of positive measures there is an element g such that $g^{-1}A$ meets B in a set of positive measure, provided that the measures m, m satisfy the condition* M_1;

(b) $m(Z)=0$ *implies $n(g^{-1}Z)=0$ for almost all elements g of G, provided that n is a measure such that n and m satisfy the condition* M_1.

PROOF. If m is substituted for n in (3.1), then the left side of (3.1) is positive. Therefore the integrand of the right side can not vanish identically, which is the assertion (a). If Z is substituted for A and G for B in (3.1), then both sides of (3.1) are zero, which implies the assertion (b).

4. Auxiliary theorem and proof of Theorem 1.

AUXILIARY THEOREM. *Let m be a measure such that $m(Xg)$ is an absolutely continuous function of the measurable set X for every g in G. Let A and B be measurable sets and let B be of finite measure. Then one of these two sets is almost congruent by finite or denumerable partition with some subset of the other.*

PROOF. Assume that no subset of one of these sets is almost congruent by finite partition with the other and that disjoint subsets A_k of A and elements g_k of G were chosen so that the sets $g_k^{-1}A_k=B_k$ are disjoint subsets of B for $k=1, 2, \cdots, n$.

Then both sets $A-\bigcup_{k=1}^n A_k$, $B-\bigcup_{k=1}^n B_k$ are of positive measure. Therefore the function

$$F_{n+1}(g) = m\left[g^{-1}\left(A - \bigcup_{k=1}^{n} A_k\right)\cap\left(B - \bigcup_{k=1}^{n} B_k\right)\right]$$

has, by Lemma 2(a), a positive upper bound M_{n+1}. We define g_{n+1} to be any element g in G with $F_{n+1}(g)>M_{n+1}/2$ and put

$$B_{n+1} = g_{n+1}^{-1}\left(A - \bigcup_{k=1}^{n} A_k\right)\cap\left(B - \bigcup_{k=1}^{n} B_k\right), \qquad A_{n+1} = g_{n+1}B_{n+1}.$$

[4] Indeed, the relations (3.1) and (3.2) for the new group operation * are identical with the relations (3.2) and (3.1), respectively, for the old group operation.

Thus an infinite sequence of disjoint measurable subsets A_k of A and a corresponding sequence of elements g_k of G has been defined by induction so that the sets $g_k^{-1}A_k = B_k$ are disjoint subsets of B.

If

$$m\left(A - \bigcup_{k=1}^{\infty} A_k\right)\cdot m\left(B - \bigcup_{k=1}^{\infty} B_k\right)$$

were positive, the function

$$F(g) = m\left[g^{-1}\left(A - \bigcup_{k=1}^{\infty} A_k\right) \cap \left(B - \bigcup_{k=1}^{\infty} B_k\right)\right]$$

would have a positive upper bound M and as $F_k(g) \geqq F(g)$ and $M_k \geqq M > 0$ for every g in G, $k = 1, 2, \cdots$, the series $\sum_{k=1}^{\infty} M_k$ would be divergent. This is in contradiction to the inequalities $M_k/2 \leqq F_k(g_k) = m(B_k)$, $\sum_{k=1}^{\infty} m(B_k) \leqq m(B) < \infty$.

Hence either $A - \bigcup_{k=1}^{\infty} A_k$ or $B - \bigcup_{k=1}^{\infty} B_k$ is of measure 0, which proves the assertion.

4.1. Proof of Theorem 1. As $m(Xg)$ is an invariant measure for every fixed g in G, it can differ from $m(X)$ only by a multiplicative constant and is, therefore, an absolutely continuous function of the measurable set X.

First case: $m(A)$ is finite. The inequality $m(A) \leqq m(B)$ and the invariance of m exclude the existence of a subset A' of A almost congruent with B by either finite or denumerable partition (symbolically: $A' \approx B$) and such that $m(A') < m(A)$. Therefore there is, by the auxiliary theorem, a subset B' of B with $A \approx B'$. If $m(B) = m(A)$, then we have $m(B - B') = 0$, which implies $A \approx B$.

Second case. $m(A)$ is infinite. Then A is the union of a sequence of disjoint measurable sets A_i of finite measure, and there is a subset B_1 of B with $A_1 \approx B_1$. The measure of $B - B_1$ being infinite, there is a subset B_2 of $B - B_1$ with $A_2 \approx B_2$. By continuing this reasoning one proves the existence of a measurable subset B' of B with $A \approx B'$. As $m(B)$ is infinite too, there is also a subset A' of A with $B \approx A'$. From $A \approx B' \subset B$ and $B \approx A' \subset B$ one deduces $A \approx B$ by replacing in the proof of F. Bernstein's equivalence theorem [**3**, p. 27] the equivalence relation by the relation $\approx$. This is legitimate as the transitivity of the relation $\approx$ is not difficult to prove.

5. Proof of Theorem 2.

Necessity. See §4.

Sufficiency. Let $m(X)$ and $n(X)$ be invariant measures and $m(Xg)$ an absolutely continuous function of the measurable set X for every

g in G. Let X_0 be a measurable set with $0<m(X_0)<\infty$ and with $c=n(X_0)/m(X_0)<\infty$.

(5.1) *$n(Z)=0$ implies $n(Z)=cm(Z)$.*

In fact, by (3.1) (with $A=Z, B=G$), $m(Zx^{-1})$ vanishes for some x, which implies $m(Z)=m[(Zx^{-1})x]=0$.

(5.2) *$0<n(Z)<\infty$ and $n(Z)=dm(Z)$ implies $d\leqq c$.*

Indeed, one sees from Lemma 2(b) that $m(Z)=0$ implies $n(Z)=0$. Therefore $n(Y)$ is an absolutely continuous additive function of the measurable set $Y\subset ZUX_0$. By the theorem of Radon-Nikodym [1, p. 36] there is, therefore, a function $f(y)$ such that we have $n(Y)=\int_Y f(y)dm(y)$ for every measurable set $Y\subset ZUX_0$.

Assume that c is less than d. Then there are numbers a, $b>a$ between c and d and the sets

$$A = X_0 \cap \underset{y}{E}\,[f(y) \leqq a], \qquad B = Z \cap \underset{y}{E}\,[f(y) \geqq b]$$

are of positive measure. Therefore, there is (by Lemma 2(a)) an element g of G such that $m(g^{-1}A\cap B)>0$. As $g^{-1}A\cap B=C$ is a subset of B, we have

$$n(C) = \int_C f(y)dm(y) \geqq bm(C).$$

On the other hand, gC being a subset of A, we have

$$n(gC) = \int_{gC} f(y)\,dm(y) \leqq am(gC) = am(C) < bm(C).$$

Hence $n(gC)<n(C)$, although n is invariant.

(5.3) *$0<n(Z)<\infty$ implies $n(Z)=cm(Z)$.*

Indeed, by (5.2) the relations $n(X_0)=cm(X_0)$, $n(Z)=dm(Z)$ imply $d\leqq c$. By interchanging Z and X_0 we get $c=d$.

Finally, if Z is an arbitrary measurable set, it is the union of a sequence of disjoint sets Z_i, with $n(Z_i)<\infty$. By (5.1) and (5.3) we have $n(Z_i)=cm(Z_i)$ hence $n(Z)=cm(Z)$, which was to be proved.

6. Proof of Theorems 3 and 3*. Extension to transformation groups.

(6.1) *If the measure m satisfies the condition M_2 resp. M_2^*, then $A\in S$ implies the relation*

$$\int_G m(xA^{-1})dm(x) = \int_G m(yA)dm(y) \tag{6.2}$$

resp.

$$(6.2^*)\qquad \int_G m(A^{-1}x)dm(x) = \int_G m(Ay)dm(y).$$

Indeed, let H be the image of the set $A \times G$ by the transformation $[(x, y)\to(yx, y)]$ resp. $[(x, y)\to(xy, y)]$. The evaluation of the integral of the characteristic function of H over $G\times G$ by the generalized theorem of Fubini furnishes the above relations.

If m is an invariant measure which satisfies the condition M_2, then $m(A)=0$ implies $m(A^{-1})=0$ on account of (6.2). Hence we have $m(g^{-1}A^{-1})=m(A^{-1})=0$ and $m(Ag)=m[(g^{-1}A^{-1})^{-1}]=0$ for every g in G, which—according to Theorem 2—proves Theorem 3.

If m and n are two invariant measures, which both satisfy the condition M_2^*, then $A\in S$ implies $A^{-1}\in S$ (by (6.2*) and the fundamental assumption).[5] Then the functions of A, $m^*(A)=m(A^{-1})$, $n^*(A)=n(A^{-1})$, are readily seen to be measures invariant under the group operation $x^*y=yx$. As M_2^* is the condition M_2 for this group operation, n^* differs from m^* only by a multiplicative constant. Therefore n differs from m by the same constant.

(6.3) The above results can be extended to measures invariant under any transitive group G of transformations operating on a measure space M, by inducing either a measure structure in G or a group structure in M.

To do this one denotes by t_x the transformation which sends a fixed element x_0 in M into x in M. Then the images $t_x\{A\}$ of the measurable subsets A of M by the mapping t_x of M onto G form a σ-field, on which one defines a measure μ by the relation

$$\mu(t_x\{A\}) = m(A).^{6}$$

A group structure can be induced in M by putting either $xy=t_xy$ or $xy=t_yx$.

References

1. A. Weil, *L'Integration dans les groupes topologiques et ses applications*, Hermann, Paris, 1938.
2. S. Saks, *Theory of the integral*, Warsaw-Lwów, 1937.
3. F. Hausdorff, *Mengenlehre*, Dover Publications, New York, 1944.
4. H. Schaerf, *Measure invariant under equivalence relation*. In preparation.

WASHINGTON UNIVERSITY

[5] By (6.2*), $A^{-1}x$ is in S for almost all x, hence $A^{-1}=(A^{-1}x)x^{-1}$ is in S too.

[6] This is a generalization of an oral remark of Prof. H. Hadwiger, Bern, communicated to the author by Prof. H. Hopf, Zurich.