A NOTE ON CONVERGENCE IN LENGTH

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1. Introduction. Let I be a closed linear interval $a_0 \leqq t \leqq b_0$. Let $\mathfrak{x}(t) = (x(t), y(t), z(t))$, $t \in I$, represent a vector function whose three components $x(t)$, $y(t)$, $z(t)$ are of bounded variation and continuous on I. This vector function determines in Euclidean 3-space a curve $x = x(t)$, $y = y(t)$, $z = z(t)$ whose length we denote by $L(\mathfrak{x})$. By convergence in length of a sequence of such vector functions $\mathfrak{x}_n(t) = (x_n(t), y_n(t), z_n(t))$, $n = 0, 1, 2, \cdots$, is meant that $x_n(t)$, $y_n(t)$, $z_n(t)$ converge uniformly on I to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively and that $L(\mathfrak{x}_n)$ converges to $L(\mathfrak{x}_0)$. We denote by $V(f)$ the total variation on I of a scalar function $f(t)$ which is continuous and of bounded variation on I. By convergence in variation of a sequence $f_n(t)$, $n = 0, 1, \cdots$, is meant that $f_n(t)$ is continuous and of bounded variation on I for $n = 0, 1, \cdots$, that $f_n(t)$ converges uniformly on I to $f_0(t)$, and that $V(f_n) \to V(f_0)$. These concepts are due to Adams, Clarkson, and Lewy [1, 2].[1]

We are concerned here with the problem of determining conditions under which convergence in length holds. Uniform convergence on I of the components $x_n(t)$, $y_n(t)$, $z_n(t)$ to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively implies only that $\liminf L(\mathfrak{x}_n) \geqq L(\mathfrak{x}_0)$. It is also well known (see [2, 4, 5]) that convergence in length of such a sequence $\mathfrak{x}_n$ implies convergence in variation of each of the three sequences of components—and, indeed, convergence in variation of any sequence of scalar functions obtained by projecting the curves $\mathfrak{x} = \mathfrak{x}_n(t)$, $t \in I$, $n = 0, 1, \cdots$, on any line whatever. As a consequence of this we see that convergence in length of the sequence $\mathfrak{x}_n(t)$ implies convergence in variation of the sequence $c_1 x_n(t) + c_2 y_n(t) + c_3 z_n(t)$ for arbitrary choice of the constants c_1, c_2, c_3. Convergence in variation of each of the three sequences of components is not sufficient to ensure convergence in length of the sequence of vectors (see [2]). In connection with the work of A. P. Morse [4] there arose the question as to whether convergence in length is implied by convergence in variation of every linear combination of the components. This has already been proved by Morse [4] for the case where $\mathfrak{x}_n(t)$ is of the special form $(t, y_n(t), 0)$, $n = 0, 1, \cdots$. In this note we generalize Morse's result to the parametric case. The proof is based on a generalization,

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

due essentially to Steinhaus [6], of a remarkable formula for arc length devised by Cauchy [3].

2. **Preliminaries.** If each component of a vector function $\mathfrak{x}(t)=(x(t), y(t), z(t))$, $t\in I$, is BV (of bounded variation), then $\mathfrak{x}(t)$ is said to be BV; if each component is AC (absolutely continuous), then $\mathfrak{x}(t)$ is said to be AC; if each component is continuous, then $\mathfrak{x}(t)$ is said to be continuous. We denote by $\mathfrak{x}'(t)$ the vector $(x'(t), y'(t), z'(t))$ wherever the derivatives $x'(t)$, $y'(t)$, $z'(t)$ all exist. If the components $x_n(t)$, $y_n(t)$, $z_n(t)$ of a sequence of vectors $\mathfrak{x}_n(t)$, $t\in I$, $n=0, 1, \cdots$, converge uniformly on I to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively, we say that $\mathfrak{x}_n(t)$ converges uniformly on I to $\mathfrak{x}_0(t)$.

Let Δ denote any subinterval $t'\leqq t\leqq t''$ contained in I and let $D(I)$ denote any subdivision of I into a finite number of nonoverlapping intervals Δ. The length $L(\mathfrak{x})$ of a vector function $\mathfrak{x}(t)$, $t\in I$, is defined as

$$L(\mathfrak{x}) = \text{l.u.b.} \sum | \mathfrak{x}(t'') - \mathfrak{x}(t') |, \qquad \Delta \in D(I),$$

where the least upper bound is taken over all subdivisions $D(I)$.

We note that this definition of length of vector functions is the exact analog of total variation of scalar functions and that it agrees with the usual definition of length of a curve if $\mathfrak{x}(t)$ is BV and continuous on I and we think of $\mathfrak{x}(t)$ as determining a curve $x=x(t)$, $y=y(t)$, $z=z(t)$, $a_0\leqq t\leqq b_0$.

We mention now the following well known facts which will be used in this note.

(a) If $\mathfrak{x}(t)=(x(t), y(t), z(t))$ is BV and continuous on I, then

$$V(x) \leqq L(\mathfrak{x}) \leqq V(x) + V(y) + V(z).$$

(b) If $f_n\to f_0(V)$, then $kf_n\to kf_0(V)$ for arbitrary choice of the constant k.

(c) If $f_n(t)$ converges uniformly on I to $f_0(t)$ and if $f_n(t)$ is BV and continuous on I for all n, then $\liminf V(f_n)\geqq V(f_0)$.

(d) If $f(t)$, $t\in I$, is BV and continuous, then $f'(t)$ is summable on I and $V(f)\geqq\int_I|f'|\,dt$, the sign of equality holding if and only if $f(t)$ is AC.

(e) If $\mathfrak{x}(t)$ is BV and continuous on I and if $\mathfrak{x}=\mathfrak{x}_n(t)$, $t\in I$, $n=1, 2, \cdots$, is a sequence of polygons inscribed in the curve $\mathfrak{x}=\mathfrak{x}(t)$, $t\in I$, and converging uniformly on I to $\mathfrak{x}=\mathfrak{x}(t)$, $t\in I$, then $\mathfrak{x}_n\to\mathfrak{x}(L)$.

(f) Let $\mathfrak{x}_n(t)=(x_n(t), y_n(t), z_n(t))$, $t\in I$, $n=0, 1, \cdots$. If $\mathfrak{x}_n\to\mathfrak{x}_0(L)$, then $x_n\to x_0(V)$.

(g) If $\mathfrak{x}(t)$, $t \in I$, is BV and continuous, then $|\mathfrak{x}'(t)|$ is summable on I and $L(\mathfrak{x}) \geqq \int_I |\mathfrak{x}'| dt$, the sign of equality holding if and only if $\mathfrak{x}(t)$ is AC.

LEMMA. *Suppose $\mathfrak{x}_n \to \mathfrak{x}_0(L)$. Let $\mathfrak{u}$ be any fixed unit vector and $f_n(t)$ the scalar product of the vectors $\mathfrak{x}_n(t)$ and $\mathfrak{u}$; that is,*

$$f_n(t) = \mathfrak{x}_n(t) \cdot \mathfrak{u}, \qquad t \in I,\ n = 0, 1, \cdots .$$

Then $f_n \to f_0(V)$ and $V(f_n) \leqq L(\mathfrak{x}_n)$ for $n = 0, 1, \cdots$.

The function $f_n(t)$ defined here is the projection of the curve $\mathfrak{x} = \mathfrak{x}_n(t)$ on a line parallel to the given vector $\mathfrak{u}$. This fact is well known, as was mentioned in §1, but a proof will be included for the convenience of the reader. Let $\mathfrak{u}$ be the vector (a, b, c), $a^2+b^2+c^2=1$. Then $f_n(t) = ax_n(t)+by_n(t)+cz_n(t)$, $n=0, 1, \cdots$. Let us set up a new system of rectangular coordinates x^*, y^*, z^*, such that the x^*-axis coincides with the line through the origin with direction cosines a, b, c. Then x^* is expressed in terms of the old coordinates as $ax+by+cz$. Let $\mathfrak{x}_n^*(t) = (x_n^*(t), y_n^*(t), z_n^*(t))$ denote the vector $\mathfrak{x}_n(t)$ referred to the new coordinates. Since arc length is independent of the particular coordinate system used, we have $\mathfrak{x}_n^* \to \mathfrak{x}_0^*(L)$ and hence (see (f), (a)) $x_n^* \to x_0^*(V)$ and $V(x_n^*) \leqq L(\mathfrak{x}_n^*)$. But $x_n^*(t) = ax_n(t)+by_n(t)+cz_n(t) = f_n(t)$, $t \in I$, $n=0, 1, \cdots$. That is,

$$f_n \to f_0(V) \text{ and } V(f_n) \leqq L(\mathfrak{x}_n)$$

for all n.

3. **Cauchy's formula.** The formula for arc length which is stated in Lemma 4 applies to any continuous, rectifiable curve in Euclidean 3-space. It is a direct generalization of a formula of Cauchy [3], the method of proof given here being due to Steinhaus [6]. A proof of the formula is included in this note because the reasoning involved in it is used in §4, as well as the formula itself.

LEMMA 1. *Let $\mathfrak{a}$ be a fixed vector and $\mathfrak{u}$ a variable unit vector from the center of a unit sphere to any point p on the surface S of the sphere. Then*

$$\iint_S |\mathfrak{a} \cdot \mathfrak{u}| \, d\sigma = 2\pi |\mathfrak{a}|,$$

where $d\sigma$ is the area-element on S.

PROOF. Choose rectangular coordinates so that the z-axis coincides in direction with $\mathfrak{a}$. In terms of the spherical coordinates $1, \theta, \phi$ of the point p on S, we have $\mathfrak{a} \cdot \mathfrak{u} = |\mathfrak{a}| \cdot \cos \phi$ and hence

$$\iint_S |\mathfrak{a}\cdot\mathfrak{u}|\, d\sigma = \int_0^{\pi}\int_0^{2\pi} |\mathfrak{a}| \cdot |\cos\phi| \sin\phi\, d\theta d\phi$$

$$= 2\pi|\mathfrak{a}|\int_0^{\pi} |\cos\phi|\sin\phi\, d\phi$$

$$= 2\pi|\mathfrak{a}|\int_0^{\pi/2} \sin 2\phi d\phi = 2\pi|\mathfrak{a}|.$$

LEMMA 2. *Let $\mathfrak{x}(t)$ be BV and continuous on I. Let S be the surface of a unit sphere, p any point on S, and $\mathfrak{u}$ the vector from the center of the sphere to p. For each point p on S let $V(p)$ denote the total variation of the function $f(t, p) = \mathfrak{x}(t)\cdot\mathfrak{u}$, $t \in I$. Then $V(p)$ is summable on the surface S.*

PROOF. Since $f(t, p)$ is clearly BV and continuous on I for each point p on S, $V(p)$ is defined for every p. Consider now a sequence of points p_n on S such that $p_n \to p_0$. It is easily verified that the sequence $f(t, p_n)$ converges uniformly on I to $f(t, p_0)$, from which it follows that lim inf $V(p_n) \geqq V(p_0)$ (see (c) of §2). This means that $V(p)$ is lower semi-continuous on S and hence measureable on S. From the fact that $V(p)$ is bounded on S by $L(\mathfrak{x})$ (see Lemma of §2), we conclude that $V(p)$ is summable on S.

LEMMA 3. *Given a sequence of BV, continuous vector functions $\mathfrak{x}_n(t)$, $t \in I$, $n = 0, 1, \cdots$. Let S, p, $\mathfrak{u}$ be defined as in Lemma 2. For each p on S let $V_n(p)$ denote the total variation of the function $f_n(t, p) = \mathfrak{x}_n(t)\cdot\mathfrak{u}$, $t \in I$, $n = 0, 1, \cdots$. If the sequence $f_n(t, p)$ converges in variation for every point p on S, then*

$$\iint_S V_n(p) d\sigma \to \iint_S V_0(p) d\sigma,$$

where $d\sigma$ is the area-element on S.

PROOF. By hypothesis the sequence $f_n(t, p)$ converges in variation for every point p on S. Hence

$$V_n(p) \to V_0(p), \qquad p \in S. \tag{1}$$

Let a, b, c be any three (real) constants such that $a^2+b^2+c^2=1$. Since the point (a, b, c) lies on S, we then have by (1) convergence in variation of the sequence $ax_n(t)+by_n(t)+cz_n(t)$ and therefore

$$V(ax_n + by_n + cz_n) \to V(ax_0 + by_0 + cz_0).$$

In particular,

$$(2) \qquad V(x_n) \to V(x_0), \qquad V(y_n) \to V(y_0), \qquad V(z_n) \to V(z_0).$$

Summability of each function $V_n(p)$ follows from Lemma 2; $V_n(p)$ converges on S to $V_0(p)$ by (1). In order to prove convergence of their integrals it will therefore be sufficient to show that the sequence $V_n(p)$ is bounded on S. For each point p on S we have by the Lemma 1 and (a) of §2

$$(3) \qquad V_n(p) \leqq L(\mathfrak{x}_n) \leqq V(x_n) + V(y_n) + V(z_n), \qquad n = 0, 1, \cdots.$$

But (2) implies the existence of a constant M such that

$$(4) \qquad V(x_n) \leqq M, \quad V(y_n) \leqq M, \quad V(z_n) \leqq M, \qquad n = 0, 1, \cdots.$$

Inequalities (3) and (4) establish the fact that the sequence $V_n(p)$ is bounded on S and hence, as remarked above,

$$\iint_S V_n(p)d\sigma \to \iint_S V_0(p)d\sigma.$$

LEMMA 4. *Under the hypotheses of Lemma* 2,

$$L(\mathfrak{x}) = (2\pi)^{-1} \iint_S V(p)d\sigma,$$

where $d\sigma$ *is the area-element on* S (*this is the generalized Cauchy formula, see* §3).

PROOF. Let us suppose first that $\mathfrak{x}(t)$ is AC. It is clear that for fixed p the function $f(t, p)$ is AC on I and that its derivative is equal to $\mathfrak{x}'(t)\cdot\mathfrak{u}$ except on a subset of I of measure zero. Hence (by (d) of §2)

$$(1) \qquad V(p) = \int_I |\mathfrak{x}'(t)\cdot\mathfrak{u}|\, dt.$$

But absolute continuity of $\mathfrak{x}(t)$ also implies (see (g) of §2)

$$(2) \qquad L(\mathfrak{x}) = \int_I |\mathfrak{x}'(t)|\, dt.$$

Since $V(p)$ is summable on S (see Lemma 2), we obtain by use of (1)

$$(3) \qquad \begin{aligned} \iint_S V(p)d\sigma &= \iint_S \left[\int_I |\mathfrak{x}'(t)\cdot\mathfrak{u}|\, dt\right] d\sigma \\ &= \int_I \left[\iint_S |\mathfrak{x}'(t)\cdot\mathfrak{u}|\, d\sigma\right] dt, \end{aligned}$$

where the theorem of Tonelli justifies the changes in the order of integration. From (3), Lemma 1, and (2) we conclude that

$$(2\pi)^{-1}\iint_S V(p)d\sigma = (2\pi)^{-1}\int_I \left[\iint_S |\mathfrak{x}'(t)\cdot\mathfrak{u}|\, d\sigma\right] dt$$

$$= (2\pi)^{-1}\int_I (2\pi)\,|\mathfrak{x}'(t)|\, dt = L(\mathfrak{x}).$$

Let us suppose next that $\mathfrak{x}(t)$ is merely BV and continuous on I. Define $\mathfrak{x}=\mathfrak{x}_n(t)$, $t\in I$, $n=1, 2, \cdots$, to be a sequence of polygons inscribed in the curve $\mathfrak{x}=\mathfrak{x}(t)$, $t\in I$, $\mathfrak{x}_n(t)$ converging uniformly on I to $\mathfrak{x}(t)$. By (e) of §2 we then have $\mathfrak{x}_n\to\mathfrak{x}_0(L)$ and hence

$$L(\mathfrak{x}_n)\to L(\mathfrak{x}). \tag{4}$$

By the lemma of §2 we also have convergence in variation of the sequence $f_n(t, p)=\mathfrak{x}_n(t)\cdot\mathfrak{u}$, $t\in I$, for every point p on S. Application of Lemma 3 then yields the result that

$$\iint_S V_n(p)d\sigma \to \iint_S V(p)d\sigma, \tag{5}$$

where $V_n(p)$ is defined as in Lemma 3 for $n=1, 2, \cdots$. But since each approximating function $\mathfrak{x}_n(t)$ is AC, we can express its length in the form

$$L(\mathfrak{x}_n) = (2\pi)^{-1}\iint_S V_n(p)d\sigma, \qquad n = 1, 2, \cdots.$$

In conjunction with (4) and (5) this implies

$$L(\mathfrak{x}) = \lim L(\mathfrak{x}_n) = \lim (2\pi)^{-1}\iint_S V_n(p)d\sigma = (2\pi)^{-1}\iint_S V(p)d\sigma.$$

4. **The theorem.** Let $\mathfrak{x}_n(t)=(x_n(t), y_n(t), z_n(t))$, $n=0, 1, \cdots$, be a sequence of vectors which are BV and continuous on I. Then $\mathfrak{x}_n\to\mathfrak{x}_0(L)$ if and only if the sequence $c_1x_n(t)+c_2y_n(t)+c_3z_n(t)$ converges in variation for every choice of the (real) constants c_1, c_2, c_3.

PROOF. *Sufficiency.* By hypothesis the sequence $c_1x_n(t)+c_2y_n(t)+c_3z_n(t)$ converges in variation for every choice of the constants c_1, c_2, c_3. This implies uniform convergence of $\mathfrak{x}_n(t)$ to $\mathfrak{x}_0(t)$ and also convergence in variation of $f_n(t, p)$ for every point p on S, where $f_n(t, p)$ is defined as in Lemma 3. From Lemmas 3 and 4 we then conclude that

$$\lim L(\mathfrak{x}_n) = \lim (2\pi)^{-1} \iint_S V_n(p)d\sigma = (2\pi)^{-1} \iint_S V_0(p)d\sigma = L(\mathfrak{x}_0)$$

and hence $\mathfrak{x}_n \to \mathfrak{x}_0(L)$.

Necessity. If $\mathfrak{x}_n \to \mathfrak{x}_0(L)$, the sequence $f_n(t, p)$ converges in variation for every point p on S (see Lemma of §2). Let c_1, c_2, c_3 be any three (real) constants. If $c_1 = c_2 = c_3 = 0$, then the statement is trivial. Otherwise let $\mathfrak{u}$ be a unit vector with direction cosines proportional to c_1, c_2, c_3. The desired relation now follows readily from the lemma and (b) of §2.

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