ON THE DISTRIBUTION OF THE MAXIMUM OF SUCCESSIVE CUMULATIVE SUMS OF INDEPENDENTLY BUT NOT IDENTICALLY DISTRIBUTED CHANCE VARIABLES

ABRAHAM WALD

1. **Introduction.** Let $X_1, X_2, \cdots$, and so on be a sequence of chance variables and let S_i denote the sum of the first i X's, that is,

$$S_i = X_1 + \cdots + X_i \quad (i = 1, 2, \cdots, \text{ad inf}). \tag{1.1}$$

Let M_N denote the maximum of the first N cumulative sums $S_1, \cdots, S_N$, that is,

$$M_N = \max (S_1, \cdots, S_N). \tag{1.2}$$

The distribution of M_N, in particular the limiting distribution of a suitably normalized form of M_N, has been studied by Erdös and Kac [1][1] and by the author [2] in the special case when the X's are independently distributed with identical distributions.

In this note we shall be concerned with the distribution of M_N when the X's are independent but not necessarily identically distributed. In particular, the mean and variance of X_i may be any functions of i.

In §2 lower and upper limits for M_N are obtained which yield particularly simple limits for the distribution of M_N when the X's are symmetrically distributed around zero.

In §3 the special case is considered when X_i can take only the values 1 and -1 but the probability p_i that $X_i=1$ may be any function of i. The exact probability distribution of M_N for this case is derived and expressed as the first row of a product of N matrices.

The limiting distribution of $M_N/N^{1/2}$ is treated in §4. Since the interesting limiting case arises when the mean of X_i ($i \leqq N$) is not only a function of i but also a function of N, we have to introduce a double sequence of chance variables. That is, for any N we consider a sequence of N chance variables $X_{N1}, \cdots, X_{NN}$. Let μ_{Ni} denote the mean and σ_{Ni} the standard deviation of X_{Ni}. Let, furthermore, S_{Ni} denote the sum $X_{N1} + \cdots + X_{Ni}$ and M_N the maximum of $S_{N1}, \cdots, S_{NN}$. With the help of a method used by Erdös and Kac [1], the following theorem is established in §4:

Presented to the Society, September 4, 1947; received by the editors June 27, 1947.

[1] Numbers in brackets refer to the references cited at the end of the paper.

THEOREM 1.1 *Let* $\{X_{Ni}\}$ *and* $\{X^*_{Ni}\}$ $(i=1, \cdots, N; N=1, 2, \cdots,$ *ad inf.*) *be two sequences of chance variables such that the following conditions are fulfilled*:

(a) *The X's are independently distributed.*

(b) *The sequence* $\{\sigma_{Ni}\}$ $(i=1, \cdots, N;\ N=1, 2, \cdots,$ *ad inf.*) *has a positive lower bound and a finite upper bound.*

(c) $\mu_{Ni}N^{1/2}$ *is a bounded function of i and N.*

(d) *The third absolute moment of* X_{Ni} *is a bounded function of i and N.*

(e) *The conditions* (a)–(d) *remain valid if we replace* X_{Ni} *by* X^*_{Ni}.

(f) *The equation*

$$\lim_{N=\infty}\left[\frac{\overset{*}{\mu}_{N1}+\cdots+\overset{*}{\mu}_{Nj_i}}{\overset{*2}{\sigma}_{N1}+\cdots+\overset{*2}{\sigma}_{Nj_i}} - \frac{\mu_{N1}+\cdots+\mu_{Ni}}{\sigma^2_{N1}+\cdots+\sigma^2_{Ni}}\left(\frac{\sigma^2_{N1}+\cdots+\sigma^2_{NN}}{\overset{*2}{\sigma}_{N1}+\cdots+\overset{*2}{\sigma}_{NN}}\right)^{1/2}\right]=0 \tag{1.3}$$

holds for all i and N where μ^*_{Ni} *is the mean and* σ^*_{Ni} *is the standard deviation* X^*_{Ni} *and* j_i *is the smallest positive integer for which*

$$\frac{\sigma^{*2}_{N1}+\cdots+\sigma^{*2}_{Nj_i}}{\sigma^{*2}_{N1}+\cdots+\sigma^{*2}_{NN}} \geq \frac{\sigma^2_{N1}+\cdots+\sigma^2_{Ni}}{\sigma^2_{N1}+\cdots+\sigma^2_{NN}}.$$

Let

$$\overline{M}_N = M^*_N\left(\frac{\sigma^2_{N1}+\cdots+\sigma^2_{NN}}{\overset{*2}{\sigma}_{N1}+\cdots+\overset{*2}{\sigma}_{NN}}\right)^{1/2} \tag{1.4}$$

where M^*_N *is the same function of the* X^**'s as* M_N *is of the X's. Then for any positive* ϵ *we have*

$$\liminf_{N=\infty}\ [\text{prob}\,\{M_N < cN^{1/2}\} - \text{prob}\,\{\overline{M}_N < (c-\epsilon)N^{1/2}\}] \geqq 0 \tag{1.5}$$

and

$$\liminf_{N=\infty}\ [\text{prob}\,\{\overline{M}_N < (c+\epsilon)N^{1/2}\} - \text{prob}\,\{M_N < cN^{1/2}\}] \geqq 0. \tag{1.6}$$

The following corollary is a simple consequence of Theorem 1.1:

COROLLARY 1.1. *Let* N' *be any positive integral valued and strictly increasing function of N for which prob* $\{\overline{M}_{N'} < cN'^{1/2}\}$ *converges to a limit function* $P(c)$ *at all continuity points c of* $P(c)$ *as* $N\to\infty$. *Then also*

(1.7) $$\lim_{N=\infty} \text{prob} \{M_{N'} < cN'^{1/2}\} = P(c)$$

at all continuity points c of P(c).

The validity of Corollary 1.1 can be derived from that of Theorem 1.1 as follows: Let $c=c_0$ be a continuity point of $P(c)$ and substitute N' for N in (1.5) and (1.6). For any positive ρ all limit points of prob $\{\overline{M}_{N'}<(c_0-\epsilon)N'^{1/2}\}$ and prob $\{\overline{M}_{N'}<(c_0+\epsilon)N'^{1/2}\}$ will lie in the interval $[P(c_0)-\rho, P(c_0)+\rho]$ for sufficiently small ϵ. Hence, equations (1.5) and (1.6) imply that

(1.8) $$\begin{aligned} P(c_0) - \rho &\leqq \liminf_{N=\infty} \text{prob} \{M_{N'} < c_0N'^{1/2}\} \\ &\leqq \limsup_{N=\infty} \text{prob} \{M_{N'} < c_0N'^{1/2}\} \leqq P(c_0) + \rho. \end{aligned}$$

Since (1.8) is true for any positive number ρ, Corollary 1.1 is proved.

The result in Corollary 1.1 can be expressed also by saying that for any subsequence $\{N'\}$ of $\{N\}$ for which $\overline{M}_{N'}/N'^{1/2}$ has a limiting distribution as $N\to\infty$, also $M_{N'}/N'^{1/2}$ has a limiting distribution which is equal to that of $\overline{M}_{N'}/N'^{1/2}$.

It can easily be verified that the conditions (e) and (f) can always be satisfied for chance variables X^*_{Ni} which take only the values 1 and -1 with properly chosen probabilities. Thus, the results of §3 may be used to compute

$$\text{prob} \left\{ M_N^* < N^{1/2}c \left(\frac{\overset{*2}{\sigma}_{N1} + \cdots + \overset{*2}{\sigma}_{NN}}{\sigma^2_{N1} + \cdots + \sigma^2_{NN}} \right)^{1/2} \right\}.$$

2. Derivation of upper and lower bounds for M_N. Let $X_1, \cdots, X_N$ be a set of N variables and let

(2.1) $$\tilde{X}_i = X_{N-i+1} \qquad (i = 1, 2, \cdots, N).$$

Let, furthermore,

(2.2) $$\tilde{M}_i = \max (\tilde{X}_i, \tilde{X}_i + \tilde{X}_{i-1}, \cdots, \tilde{X}_i + \cdots + \tilde{X}_1), \qquad (i = 1, \cdots, N).$$

Clearly

(2.3) $$\tilde{M}_N = M_N = \max (X_1, X_1 + X_2, \cdots, X_1 + \cdots + X_N).$$

If $X_1, \cdots, X_N$ are independent chance variables, the chance variables $\tilde{M}_1, \tilde{M}_2, \cdots, \tilde{M}_N$ form a simple Markoff chain, that is, the conditional distribution of $\tilde{M}_{i+1}$, given $\tilde{M}_1, \cdots, \tilde{M}_i$, depends only on $\tilde{M}_i$. This is an immediate consequence of the relations:

$$\tilde{M}_{i+1} = \tilde{M}_i + \tilde{X}_{i+1} \qquad \text{if } \tilde{M}_i > 0 \tag{2.4}$$

and

$$\tilde{M}_{i+1} = \tilde{X}_{i+1} \qquad \text{if } \tilde{M}_i \leq 0. \tag{2.5}$$

We shall now prove the following theorem:

THEOREM 2.1. *The inequality*

$$\tilde{M}_i \leqq |\epsilon_1\tilde{X}_1 + \cdots + \epsilon_i\tilde{X}_i| \qquad (i = 1, \cdots, N) \tag{2.6}$$

holds where $\epsilon_1 = 1$, $\epsilon_i = 1$ *if* $\epsilon_1\tilde{X}_1 + \cdots + \epsilon_{i-1}\tilde{X}_{i-1} > 0$ *and* $\epsilon_i = -1$, *if* $\epsilon_1\tilde{X}_1 + \cdots + \epsilon_{i-1}\tilde{X}_{i-1} \leqq 0$.

PROOF. Clearly, (2.6) holds for $i=1$. We shall prove (2.6) for $i+1$ assuming that it holds for i. For this purpose it is sufficient to show, because of (2.4) and (2.5), that

$$|\epsilon_1\tilde{X}_1 + \cdots + \epsilon_{i+1}\tilde{X}_{i+1}| - |\epsilon_1\tilde{X}_1 + \cdots + \epsilon_i\tilde{X}_i| \geqq \tilde{X}_{i+1}. \tag{2.7}$$

Denote $|\epsilon_1, \tilde{X}_1 + \cdots + \epsilon_i\tilde{X}_i|$ by c_i. If $c_i > 0$, then $\epsilon_{i+1} = 1$ and inequality (2.7) goes over into

$$|c_i + \tilde{X}_{i+1}| - c_i \geqq \tilde{X}_{i+1}, \tag{2.8}$$

which is obviously true. If $c_i \leqq 0$, $\epsilon_{i+1} = -1$ and inequality (2.7) is equivalent with

$$||c_i| + \tilde{X}_{i+1}| - |c_i| \geqq \tilde{X}_{i+1}, \tag{2.9}$$

which is obviously true. Hence, Theorem 2.1 is proved.

We shall now prove a theorem giving a lower bound for $\tilde{M}_i$.

THEOREM 2.2. *The inequality*

$$\tilde{K}_i = |\epsilon_1\tilde{X}_1 + \cdots + \epsilon_i\tilde{X}_i| - 2 \max_{j \leqq i} |\tilde{X}_j| \leqq \tilde{M}_i \qquad (i = 1, \cdots, N) \tag{2.10}$$

holds where the ϵ*'s are defined as in Theorem* 2.1.

PROOF. Theorem 2.2 is obviously true for $i=1$. We shall assume that it is valid for i and we shall prove it for $i+1$. It follows from (2.4) and (2.5) that

$$\tilde{M}_{i+1} - \tilde{M}_i \geqq \tilde{X}_{i+1}, \tag{2.11}$$

$$\tilde{M}_{i+1} \geqq \tilde{X}_{i+1}. \tag{2.12}$$

Hence, to prove (2.10) for $i+1$ assuming that it is true for i, it is sufficient to show that at least one of the following two inequalities holds:

$$\tilde{K}_{i+1} - \tilde{K}_i \leqq \tilde{X}_{i+1}, \tag{2.13}$$

$$\tilde{K}_{i+1} \leqq \tilde{X}_{i+1}. \tag{2.14}$$

Consider first the case when $|\tilde{X}_{i+1}| \leqq |\epsilon_1\tilde{X}_1 + \cdots + \epsilon_i\tilde{X}_i|$. In this case (2.13) always holds, as can easily be verified. If $|\tilde{X}_{i+1}| > |\epsilon_1\tilde{X}_1 + \cdots + \epsilon_i\tilde{X}_i|$ and $\tilde{X}_{i+1} \geqq 0$, then (2.13) holds again. If $|\tilde{X}_{i+1}| > |\epsilon_1\tilde{X}_1 + \cdots + \epsilon_i\tilde{X}_i|$ and $\tilde{X}_{i+1} < 0$, then $|\epsilon_1\tilde{X}_1 + \cdots + \epsilon_i\tilde{X}_i + \epsilon_{i+1}\tilde{X}_{i+1}| \leqq |\tilde{X}_{i+1}|$ and, therefore, $\tilde{K}_{i+1} \leqq |\tilde{X}_{i+1}| - 2 \max_{j \leqq i+1} |X_j| \leqq -|\tilde{X}_{i+1}| = \tilde{X}_{i+1}$. Thus, in this case the inequality (2.14) holds. This completes the proof of Theorem 2.2.

Since $\tilde{M}_N = M_N$, Theorems 2.1 and 2.2 yield the following limits for M_N

$$\begin{aligned} &|\epsilon_1\tilde{X}_1 + \cdots + \epsilon_N\tilde{X}_N| - 2 \max_{i \leqq N} |\tilde{X}_i| \\ &\qquad \leqq M_N \leqq |\epsilon_1\tilde{X}_1 + \cdots + \epsilon_N\tilde{X}_N|. \end{aligned} \tag{2.15}$$

Suppose now that $X_1, \cdots, X_N$ are chance variables such that the conditional distribution of X_i ($i=1, \cdots, N$) for any given values of $X_{i+1}, \cdots, X_N$ is symmetric around the origin. Then the probability distribution of $|\epsilon_1\tilde{X}_1 + \cdots + \epsilon_N\tilde{X}_N|$ is the same as that of $|X_1 + \cdots + X_N|$, and the distribution of $|\epsilon_1\tilde{X}_1 + \cdots + \epsilon_N\tilde{X}_N| - 2 \max_{i \leqq N} |\tilde{X}_i|$ equals that of $|X_1 + \cdots + X_N - 2\max_{i \leqq N}|X_i||$. It then follows from (2.15) that the following theorem holds:

THEOREM 2.3. *If the conditional distribution of X_i ($i=1, 2, \cdots, N$), for any given value of $X_{i+1}, \cdots, X_N$ is symmetric around the origin, the inequality*

$$\begin{aligned} &\text{prob}\{|X_1 + \cdots + X_N| < c\} \leqq \text{prob}\{M_N < c\} \\ &\qquad \leqq \text{prob}\{|X_1 + \cdots + X_N| - 2 \max_{i \leqq N} |X_i| < c\} \end{aligned} \tag{2.16}$$

holds for any value c.

Inequality (2.15) has also some interesting implications for the asymptotic distribution theory of M_N. In most cases we shall be concerned with the limiting distribution of $M_N/N^{1/2}$ as $N \to \infty$ (this is the case discussed in §4). If $(1/N^{1/2}) \max_{i \leqq N} |X_i|$ converges stochastically to zero, as will usually be the case, inequality (2.15) implies that the limiting distribution of $M_N/N^{1/2}$ is the same as that of $(1/N^{1/2})|\epsilon_1\tilde{X}_1 + \cdots + \epsilon_N\tilde{X}_N|$.

3. The distribution of M_N when X_i can take only the values 1 and -1. Let $X_1, \cdots, X_N$ be independently distributed chance variables such that X_i can take only the values 1 and -1. Let p_i denote the probability that $X_i=1$. The probability that $X_i=-1$ is then equal to $1-p_i=q_i$.

Let $\tilde{X}_i$ and $\tilde{M}_i$ $(i=1, \cdots, N)$ be defined by (2.1) and (2.2), respectively. One can easily verify that $\tilde{M}_i$ can take only the values $-1, 0, 1, 2, \cdots, i$. Let c_{ij} denote the probability that $\tilde{M}_i=j$ for $j=1, \cdots, i$, and let c_{i0} be the probability that $\tilde{M}_i \leqq 0$. It follows from the definition of the $\tilde{M}$'s that the following recursion formulas hold:

$$c_{i+1,0} = q_{i+1}c_{i0} + q_{i+1}c_{i1}, \tag{3.1}$$

$$c_{i+1,j} = p_{i+1}c_{i,j-1} + q_{i+1}c_{i,j+1} \qquad (j = 1, 2, \cdots, i+1). \tag{3.2}$$

Since $\tilde{M}_N = M_N$, we have

$$\text{prob}\,\{M_N = j\} = c_{Nj} \qquad \text{for } j = 1, \cdots, N, \tag{3.3}$$

$$\text{prob}\,\{M_N \leqq 0\} = c_{N0}. \tag{3.4}$$

We shall now construct N square matrices $A_1, \cdots, A_N$, each having $N+1$ rows and $N+1$ columns, such that the first row of the product matrix $A_1A_2 \cdots A_N$ is equal to $(c_{N0}, c_{N1}, \cdots, c_{NN})$. Let $\overset{k}{a}_{ij}$ denote the element in the ith row and jth column of the matrix A_k $(i, j=1, \cdots, N+1;\ k=1, \cdots, N)$. We put

$$\begin{aligned} \overset{k}{a}_{11} = q_k; \qquad \overset{k}{a}_{i,i+1} = p_k \qquad (i = 1, 2, \cdots, N); \\ \overset{k}{a}_{i,i-1} = q_k \qquad (i = 2, 3, \cdots, N+1) \end{aligned} \tag{3.5}$$

and all other elements $\overset{k}{a}_{ij}$ equal to zero. It then follows easily from the recursion formulas (3.1) and (3.2) that the first row of the product $A_1A_2 \cdots A_N$ is equal to $(c_{N0}, c_{N1}, \cdots, c_{NN})$. Thus, the first row of the product $A_1A_2 \cdots A_N$ yields the exact probability distribution of M_N.

Starting with the initial values $c_{10}=q_1$, $c_{11}=p_1$, $c_{1j}=0$ for $j>1$, the final values $c_{N0}, c_{N1}, \cdots, c_{NN}$ can be best computed by repeated application of the recursion formulas (3.1) and (3.2).

4. Proof of Theorem 1.1. Let $\{X_{Ni}\}$ and $\{X^*_{Ni}\}$ be two double sequences of chance variables for which conditions (a)–(f) of Theorem 1.1 are fulfilled. Let k be a positive integer and $N_1, \cdots, N_k$ a set of positive integers such that $N_1<N_2< \cdots <N_k=N$. Let, furthermore,

$$(4.1)\qquad P_{N,k}(c) = \text{prob}\,\{\max (S_{NN_1}, S_{NN_2}, \cdots, S_{NN_k}) < cN^{1/2}\}.$$

Because of conditions (b) and (c) of Theorem 1.1, there exist two finite values A and B such that $A \geqq N\mu_{Ni}^2$ and $B \geqq \sigma_{Ni}^2$ for all N and i. Let $\phi(k)$ be an upper bound of the values

$$(4.2)\qquad \frac{N_1}{N}, \quad \frac{N_2 - N_1}{N}, \cdots, \quad \frac{N_k - N_{k-1}}{N}.$$

For any positive ϵ the following inequality holds:

$$(4.3)\qquad P_{N,k}(c - \epsilon) - \frac{\phi(k)}{\epsilon^2}\,[B + A\phi(k)] \leqq P_N(c) \leqq P_{N,k}(c),$$

where $P_N(c) = \text{prob}\,\{M_N < cN^{1/2}\}$. Using a method given by Erdös and Kac [1], the author [2] has proved the above inequality when $\mu_{Ni} = \mu_N$, $\sigma_{Ni} = 1$ and $N_j = [jN/k]$. To adapt the proof given in [2] to the more general case treated here, it is sufficient to replace the right-hand member of (2.6) in [2] by

$$(4.4)\qquad \frac{(N_{i+1} - N_i)B + (N_{i+1} - N_i)^2 \mu_N^2}{\epsilon^2 N},$$

where $\mu_N^2 = \max (\mu_{N1}^2, \cdots, \mu_{NN}^2)$.

For the purpose of proving Theorem 1.1, we shall choose N_j to be the smallest positive integer for which

$$(4.5)\qquad \sigma_{N1}^2 + \cdots + \sigma_{NN_j}^2 \geqq \frac{j(\sigma_{N1}^2 + \cdots + \sigma_{NN}^2)}{k}.$$

Since σ_{Ni}^2 has a positive lower bound and a finite upper bound, there exists a positive constant h, independent of k, such that h/k is an upper bound of the values (4.2). It then follows from (4.3) that

$$(4.6)\qquad P_{N,k}(c - \epsilon) - \frac{1}{\epsilon^2 k}\,(a + b/k) \leqq P_N(c) \leqq P_{N,k}(c)$$

when a and b are positive constants independent of N, k, c and ϵ.

Clearly, if Theorem 1.1 is true for the special case when $\sigma_{N1}^2 + \cdots + \sigma_{NN}^2 = \sigma_{N1}^{*2} + \cdots + \sigma_{NN}^{*2}$, it must be true also in the general case. Hence, it is sufficient to prove Theorem 1.1 when $\sigma_{N1}^2 + \cdots + \sigma_{NN}^2 = \sigma_{N1}^{*2} + \cdots + \sigma_{NN}^{*2}$. In what follows we shall therefore restrict ourselves to this special case.

Let N_j^*, $P_{N,k}^*(c)$, and $P_N^*(c)$ have the same meaning with reference to the X^*'s as N, $P_{N,k}(c)$, and $P_N(c)$ with reference to the X's. Then we have

$$(4.7)\qquad P^*_{N,k}(c-\epsilon) - \frac{1}{\epsilon^2 k}(a^* + b^*/k) \leqq P^*_N(c) \leqq P^*_{N,k}(c),$$

where a^* and b^* are positive constants independent of N, k, c and ϵ.

Let $G^N_{k1}, G^N_{k2}, \cdots, G^N_{kk}$ be independently and normally distributed chance variables and let the mean and standard deviation of G^N_{ki} be equal to the mean and standard deviation of $(k/N)^{1/2}(S_{NN_i} - S_{NN_{i-1}})$, respectively. Let, furthermore,

$$(4.8)\qquad \begin{aligned} &Q_{N,k}(c) \\ &= \text{prob}\left\{\max\left(G^N_{k1}, G^N_{k1} + G^N_{k2}, \cdots, G^N_{k1} + \cdots + G^N_{kk}\right) < ck^{1/2}\right\}. \end{aligned}$$

Clearly, the mean and standard deviation of G^N_{ki} are bounded functions of N, k and i. Furthermore, the standard deviation of G^N_{ki} has a positive lower bound. It then follows from condition (d) and the central limit theorem that

$$(4.9)\qquad \lim_{N=\infty}\left[Q_{N,k}(c) - P_{N,k}(c)\right] = 0.$$

Let G^{*N}_{ki} and $Q^*_{N,k}(c)$ have the same meaning with reference to the X^*'s as G^N_{ki} and $Q_{N,k}(c)$ with reference to the X's. We then have

$$(4.10)\qquad \lim_{N=\infty}\left[Q^*_{N,k}(c) - P^*_{N,k}(c)\right] = 0.$$

It follows from condition (f) of Theorem 1.1 that

$$(4.11)\qquad \lim_{N=\infty} E(G^N_{ki} - G^{*N}_{ki}) = 0,$$

$$(4.12)\qquad \lim_{N=\infty} E\left[(G^N_{ki})^2 - (G^{*N}_{ki})^2\right] = 0.$$

Hence

$$(4.13)\qquad \lim_{N=\infty}\left[Q_{N,k}(c) - Q^*_{N,k}(c)\right] = 0.$$

From (4.9) and (4.10) and (4.13) we obtain

$$(4.14)\qquad \lim_{N=\infty}\left[P_{N,k}(c) - P^*_{N,k}(c)\right] = 0.$$

Equations (4.6) and (4.14) give

$$(4.15)\qquad \liminf_{N=\infty}\left[P_N(c) - P^*_{N,k}(c-\epsilon) + \frac{1}{\epsilon^2 k}\left(a + \frac{b}{k}\right)\right] \geqq 0$$

and

$$\liminf_{N=\infty} [P^*_{N,k}(c) - P_N(c)] \geqq 0. \tag{4.16}$$

Since

$$P^*_{N,k}(c - \epsilon) \geqq P^*_N(c - \epsilon) \tag{4.17}$$

and since, because of (4.7),

$$P^*_{N,k}(c) - \frac{1}{\epsilon^2 k}(a^* + b^*/k) \leqq P^*_N(c + \epsilon), \tag{4.18}$$

we obtain from (4.15) and (4.16)

$$\liminf_{N=\infty} \left[P_N(c) - P^*_N(c - \epsilon) + \frac{1}{\epsilon^2 k}\left(a + \frac{b}{k}\right)\right] \geqq 0 \tag{4.19}$$

and

$$\liminf_{N=\infty} \left[P^*_N(c + \epsilon) + \frac{1}{\epsilon^2 k}\left(a^* + \frac{b^*}{k}\right) - P_N(c)\right] \geqq 0. \tag{4.20}$$

Hence, since k can be chosen arbitrarily large, we obtain

$$\liminf_{N=\infty} [P_N(c) - P^*_N(c - \epsilon)] \geqq 0 \tag{4.21}$$

and

$$\liminf_{N=\infty} [P^*_N(c + \epsilon) - P_N(c)] \geqq 0. \tag{4.22}$$

This concludes the proof of Theorem 1.1. It may be of interest to note that (4.21) and (4.22) imply that for any subsequence $\{N'\}$ of the sequence $\{N\}$ we have

$$\begin{aligned} \liminf_{N=\infty} P^*_{N'}(c - \epsilon) &\leqq \liminf_{N=\infty} P_{N'}(c) \leqq \limsup_{N=\infty} P_{N'}(c) \\ &\leqq \limsup_{N=\infty} P^*_{N'}(c + \epsilon). \end{aligned} \tag{4.23}$$

References

1. P. Erdös and M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 292–302.

2. A. Wald, *Limit distribution of the maximum and minimum of successive cumulative sums of random variables*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 142–153.

Columbia University