RECURSIVE PROPERTIES OF TRANSFORMATION GROUPS. II

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The purpose of this note is to sharpen a previous result on the transmission of recursive properties of a transformation group to certain of its subgroups. [See *Recursive properties of transformation groups*, by W. H. Gottschalk and G. A. Hedlund, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 637–641.]

Let T be a multiplicative topological group with identity e. A subset R of T is said to be *relatively dense* provided that $T=RK$ for some compact set K in T.

LEMMA 1. *If R is a relatively dense closed semi-group ($RR\subset R$) in T, then R is a subgroup of T.*

PROOF. Suppose $r\in R$ and U is a neighborhood of e. It is sufficient to show that $r^{-1}U\cap R\neq\emptyset$. Let V be a neighborhood of e for which $VV^{-1}\subset U$ and let K be a compact set in T for which $T=RK$. There exists a finite collection F of right translates of V which covers K. Choose $k_0\in K$. Now $r^{-1}k_0=r_1k_1$ for some $r_1\in R$ and some $k_1\in K$. Again $r^{-1}k_1=r_2k_2$ for some $r_2\in R$ and some $k_2\in K$. This may be continued. Thus there exist sequences $k_0, k_1, \cdots$ in K and $r_1, r_2, \cdots$ in R such that $r^{-1}k_i=r_{i+1}k_{i+1}$ $(i=0, 1, \cdots)$. Select integers m and n $(0\leqq m<n)$ and an element V_0 of F such that $k_m, k_n\in V_0$. Now $r^{-1}k_mk_n^{-1} = (r^{-1}k_mk_{m+1}^{-1})(k_{m+1}k_{m+2}^{-1})\cdots(k_{n-1}k_n^{-1}) = r_{m+1}rr_{m+2}\cdots rr_n\in R$. Also $r^{-1}k_mk_n^{-1}\in r^{-1}V_0V_0^{-1}\subset r^{-1}VV^{-1}\subset r^{-1}U$. Hence $r^{-1}U\cap R\neq\emptyset$ and the proof is completed.

Now let T act as a transformation group on a topological space X. That is to say, suppose that to $x\in X$ and $t\in T$ is assigned a point, denoted xt, of X such that: (1) $xe=x$ $(x\in X)$; (2) $(xt)s=x(ts)$ $(x\in X;\, t, s\in T)$; (3) The function xt defines a continuous transformation of $X\times T$ into X. We assume for the remainder of the paper that x is a fixed point of X, T is locally compact and S is a relatively dense invariant subgroup of T. Let Σ denote the maximal subset of T for which $x\Sigma\subset(xS)^*$ where the star denotes the closure operator.

LEMMA 2. *The set Σ is a closed subgroup of T which contains S.*

PROOF. Obviously $\Sigma\supset S$. From $x\Sigma^*\subset(x\Sigma)^*\subset(xS)^*$ we conclude that Σ is closed. By Lemma 1 it is now enough to show that Σ is a

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semi-group. Suppose σ, $\tau \in \Sigma$. From $x\sigma \in (xS)^*$ it follows that $x\sigma\tau \in (xS)^*\tau \subset (xS\tau)^* \subset (x\tau S)^*$. From $x\tau \in (xS)^*$ it follows that $x\tau S \subset (xS)^*S \subset (xSS)^* \subset (xS)^*$. Hence $x\sigma\tau \in (xS)^*$. Thus $\sigma\tau \in \Sigma$ and the proof is completed.

LEMMA 3. *If W is a neighborhood of e, then $x \notin (x[T-\Sigma W])^*$.*

PROOF. We first show that if $t \in T-\Sigma$, then $x \notin (x\Sigma V_0)^*$ for some neighborhood V_0 of t. Suppose $t \in T-\Sigma$. Since $t^{-1} \notin \Sigma$ by Lemma 2, $xt^{-1} \notin (x\Sigma)^*$ and $x \notin (x\Sigma t)^*$. There are neighborhoods U of x and V of e such that $V = V^{-1}$ and $UV \cap x\Sigma t = \emptyset$. It follows that $U \cap x\Sigma tV = \emptyset$. Define $V_0 = tV$.

We may assume W is open. Define $N = K - \Sigma W$ where K is a compact set in T such that $T = SK$. Using Lemma 2 we conclude that $T = SK \subset S(N \cup \Sigma W) \subset SN \cup S\Sigma W \subset \Sigma N \cup \Sigma W$ and $\Sigma N \cap \Sigma W = \emptyset$. Hence $T - \Sigma W = \Sigma N$. By the preceding paragraph, to each $n \in N$ there corresponds a neighborhood V_n of n such that $x \notin (x\Sigma V_n)^*$. Since finitely many of the V_n cover N, $x \notin (x\Sigma N)^*$. The proof is completed.

LEMMA 4. *If U is a neighborhood of x, then there exists a compact set M in T such that $xM \subset U$ and $\Sigma \subset SM^{-1}$.*

PROOF. Define $N = K \cap \Sigma$ where K is a compact set in T such that $T = SK$. If $n \in N$, then $xn \in (xS)^*$ and $x \in (xSn^{-1})^*$. Thus $n \in N$ implies the existence of $s_n \in S$ such that $xs_nn^{-1} \in \operatorname{int} U$ and hence the existence of a compact neighborhood W_n of s_nn^{-1} such that $xW_n \subset U$. Since N is compact by Lemma 2, there is a finite subset F of N for which $N \subset \bigcup_{n\in F} W_n^{-1}s_n$. Define $M = \bigcup_{n \in F} W_n$. Clearly $xM \subset U$. Using Lemma 2 we conclude that $\Sigma \subset SN \subset SM^{-1}$. The proof is completed.

Let there be distinguished in T certain sets, called *admissible*, which satisfy this condition: If A is an admissible set and if B is a set in T such that $A \subset BK$ for some compact set K in T, then B is an admissible set. A subgroup R of T is said to be *recursive* at x provided that to each neighborhood U of x there corresponds an admissible set A such that $A \subset R$ and $xA \subset U$.

LEMMA 5. *If T is recursive at x, then Σ is recursive at x.*

PROOF. Let U be a neighborhood of x. There are neighborhoods V of x and W of e such that $W = W^{-1}$, W is compact and $VW \subset U$. By Lemma 3 we may suppose that $V \cap x(T - \Sigma W) = \emptyset$. There exists an admissible set A in T such that $xA \subset V$. Clearly $A \subset \Sigma W$ and $xAW \subset U$. Define $B = \Sigma \cap AW$. Since $A \subset BW$, B is an admissible set. Also $B \subset \Sigma$ and $xB \subset U$. The proof is completed.

LEMMA 6. *If Σ is recursive at x, then S is recursive at x.*

PROOF. Let U be an open neighborhood of x. By Lemma 4 there exists a compact set M in T such that $xM \subset U$ and $\Sigma \subset SM^{-1}$. Let V be a neighborhood of x for which $VM \subset U$. There exists an admissible set A such that $A \subset \Sigma$ and $xA \subset V$. Hence $xAM \subset U$. Define $B = S \cap AM$. Since $A \subset BM^{-1}$, B is an admissible set. Also $B \subset S$ and $xB \subset U$. The proof is completed.

The following theorem is an immediate consequence of Lemmas 5 and 6.

THEOREM. *If T is recursive at x, then S is recursive at x.*

An interpretation of admissibility arises if we define an admissible subset of T to be a relatively dense subset of T. The term "recursive" is then replaced by "almost periodic." For other applications, see the paper cited above.

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FIXED POINT THEOREMS FOR INTERIOR TRANSFORMATIONS

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If M is a bounded continuum in a Euclidean plane E which does not separate E and T is an interior continuous transformation of M onto a subset of E which contains M, does T leave a point of M invariant? It is the purpose of this paper to answer this question in the affirmative for certain types of locally connected continua.

Using a notation introduced by Eilenberg [**2**, p. 168][1] a continuum M will be said to have property (b) provided every continuous transformation of M into the unit circle S in the Cartesian plane, with center at o, is homotopic to a constant mapping, that is, a transformation which transforms each point of M into a single point of S. If T is a continuous transformation of a subset A of the plane E into a subset B of E, then for each point x of A let $T'(x)$ be the point y of S such that the directed line segment oy is parallel in direction and sense to the directed line segment x, $T(x)$. Then T' will be referred to as the transformation of A into S derived from T. Such a transformation

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[1] Numbers in brackets refer to the references cited at the end of the paper.