$$b_1 = b_2 = \cdots = b_n = 0, \beta = 0, \rho_2 = 0, \epsilon = 1 \qquad \text{of V and VI,}$$

and

$$a_1 = a_2 = \cdots = a_n = 0, \alpha = 0, \rho = 0, e = 1, \lambda = 1 \qquad \text{of VII,}$$

are due to G. Szegö.

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# SOME REMARKS ON POLYNOMIALS

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This note contains some disconnected remarks on polynomials.

Let $f_n(x)=\prod_{i=1}^{n}(x-x_i)$, $-1\leqq x_1\leqq x_2\leqq \cdots \leqq x_n\leqq 1$. Denote by $-1\leqq y_1\leqq \cdots \leqq y_{n-1}\leqq 1$ the roots of $f'_n(x)$. We prove the following theorem.

THEOREM 1. *For all $n$*

$$(1) \qquad |f_n(-1)| + |f_n(+1)| + \sum_{i=1}^{n-1} |f_n(y_i)| \leqq 2^n.$$

*For $n\geqq 3$*

$$(2) \qquad |f_n(-1)|^{1/2} + |f_n(+1)|^{1/2} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/2} \leqq 2^{n/2}.$$

*For $n\geqq n_0(k)$*

$$(3) \qquad |f_n(-1)|^{1/k} + |f_n(+1)|^{1/k} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/k} \leqq 2^{n/k}.$$

REMARK. If $y_i=y_{i+1}$ or $-1=y_1$, $+1=y_{n-1}$ the corresponding summands clearly vanish.

Clearly

$$|f_n(-1)| \leqq (1-x_1)2^{n-1}, \qquad |f_n(y_i)| \leqq |y_i - x_{i+1}|\, 2^{n-1},$$
$$|f_n(+1)| \leqq (1-x_n)2^{n-1}.$$

Thus

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Received by the editors March 28, 1947.

$$|f_n(-1)| + |f_n(+1)| + \sum_{i=1}^{n-1} |f_n(y_i)|$$
$$\leqq \left((1 - x_1) + \sum_{i=1}^{n-1} (y_i - x_{i+1}) + (1 - x_n)\right) 2^{n-1} \leqq 2^n,$$

which proves (1)

We have by the inequality of the geometric and arithmetic mean

$$|f_n(-1)|^{1/2} \leqq \frac{(1 - x_1) + (1 - x_2)}{2} 2^{n/2-1},$$
$$|f_n(y_i)|^{1/2} \leqq \frac{(x_{i+1} - x_i)}{2} 2^{n/2-1},$$
$$|f_n(+1)|^{1/2} \leqq \frac{(1 - x_n) + (1 - x_{n-1})}{2} 2^{n/2-1}.$$

Thus we evidently have for $n \geqq 3$

$$|f_n(-1)|^{1/2} + |f_n(+1)|^{1/2} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/2} \leqq 2^{n/2},$$

which proves (2).

$f_1(x) = x$ and $f_2(x) = x^2/2 - 1$ shows that (2) is false for $n < 3$. Clearly equality occurs in (1) and (2) only for $\pm(1 \pm X)^n$.

The proof of (3) is more complicated and since the proof does not present any particular interest we are just going to sketch it. Let $f_n(x)$ be the polynomial which maximizes the sum (3). If (3) is not true we must have

$$|f_n(x_0)| = \max_{-1 \leqq x \leqq 1} |f_n(x)| > \frac{2^n}{n^k}\cdot$$

But then it is easy to see that $x_0$ does not lie in $(1-\epsilon, -1+\epsilon)$; without loss of generality we can assume that $1-\epsilon < x_0 \leqq 1$, and $n+o(n)$ of the $x_i$ are in $(-1+\delta, -1)$. But then a simple computation shows that $f_n(x)$ has no roots in $(1-\epsilon, 1)$ and thus $x_0 = 1$. (This is clear since if we move any possible root of $f_n(x)$ in $(1-\epsilon, 1)$ to $-1$, we clearly increase the sum (3).) By the same argument we obtain by a simple calculation that all the roots of $f_n(x)$ have to be in $-1$, which proves (3) and completes the proof of Theorem 1.

At present I can not determine the exact value of $n_0(k)$.

Let $g_n(z) = \prod_{i=1}^{n} (z - z_i)$, $|z_i| = 1$. Denote by $u_1, u_2, \cdots$ the local extremal points of $g_n(z)$, that is, the points where the vector $g_n'(z)$

points either towards the origin or away from it. I conjectured that

$$\sum_i |g_n(u_i)| \leqq 2^n.$$

Professor Breusch[1] proved this conjecture for sufficiently large $n$. The proof is complicated. For small values of $n$ he showed by examples that the result is false.

Let $-1=x_0<x_1\leqq \cdots \leqq x_n=x_{n+1}=1$. Put $\omega(x)=\prod_{i=1}^n (x-x_i)$, $l_k(x)=\omega(x)/\omega(x_k)(x-x_k)$, the fundamental functions of Lagrange interpolation. The problem of determining the set for which

$$\max_{-1\leqq x\leqq 1} \sum_{k=1}^n | l_k(x) |$$

is minimal is still unsolved. It has been conjectured, but never proved, that the minimum is attained for the points for which all the $n+1$ sums

$$\max_{x_i\leqq x\leqq x_{i+1}} \sum_{k=1}^n | l_k(x) |, \qquad i=0, 1, \cdots, n, \tag{4}$$

are equal. If the $x_i$ are the roots of $T_n(x)$ (the $n$th Tchebychef polynomial), then a simple computation shows that the sums (4) all equal

$$\frac{2}{\pi} \log n + O(1).$$

S. Bernstein[2] proved that for any $-1\leqq x_1\leqq x_2\leqq \cdots \leqq x_n\leqq 1$

$$\max_{-1\leqq x\leqq 1} \sum_{k=1}^n | l_k(x) | > (1+o(1)) \frac{2}{\pi} \log n,$$

and I proved[3] that

$$\max_{-1\leqq x\leqq 1} \sum_{k=1}^n | l_k(x) | > \frac{2}{\pi} \log n - c \qquad (c \text{ absolute constant}).$$

We consider a slightly different problem. We prove the following theorem.

THEOREM 2. *Let* $-1=x_0\leqq x_1 \cdots \leqq x_n\leqq x_{n+1}=1$. *Then for some* $i$

$$\max_{x_i<x<x_{i+1}} \sum_{k=1}^n | l_k(x) | < n^{1/2}. \tag{5}$$

[1] Oral communication.
[2] Bull. Acad. Sci. URSS. Sér. Math. (1931) pp. 1025–1050.
[3] Unpublished.

REMARK. $n^{1/2}$ in (5) can very likely be improved to $c \log n$. In fact it is likely that

$$\min_{0 \leqq x_i \leqq n} \max_{x_i \leqq x \leqq x_{i+1}} \sum_{k=1}^{n} | l_k(x) |$$

assumes its maximum when all the sums (4) are equal.

If, for some $i$, $x_i = x_{i+1}$ then (5) is obvious. Assume that $x_i \neq x_{i+1}$, $0 \leqq i \leqq n+1$. Consider the equation $\sum_{k=1}^{n} l_k^2(x) = 1$. The number of solutions is not greater than $2n-2$ and $x_1, x_2, \cdots, x_n$ are solutions. Thus a simple argument shows that for some $i$, $1 \leqq i \leqq n-1$,

$$\sum_{k=1}^{n} l_k^2(x) < 1 \qquad \text{for } x_i < x < x_{i+1}.$$

But then clearly (from Schwartz's inequality)

$$\sum_{k=1}^{n} | l_k(x) | < n^{1/2} \qquad \text{for } x_i \leqq x \leqq x_{i+1}$$

which proves Theorem 2.

In one of his interesting papers Schur[4] proves among others the following result: Let $a_0 x^n + \cdots + a_n$ be a polynomial with integer coefficients, all whose roots are in $(-1, +1)$ and are different. Then for sufficiently large $n$, $|a_0| > (2^{1/2} - \epsilon)^n$. We prove the stronger theorem:

THEOREM 3. *Let $f_n(x) = a_0 x^n + \cdots + a_n$ be a polynomial with integer coefficients and $f_n(-1) \neq 0$, $f_n(0) \neq 0$, $f_n(+1) \neq 0$. Then,* $|a_0| \geqq 2^{n/2}$.

We have $(x_1, x_2, \cdots, x_n$ are the roots of $f_n(x))$

$$| f_n(-1) f_n^2(0) f_n(+1) | = \left| a_0^4 \prod_{i=1}^{n} (1 - x_i^2) x_i^2 \right| \geqq 1.$$

But $|(1 - x_i^2) x_i^2| \leqq 1/4$. Thus $|a_0| \geqq 2^{n/2}$ which completes the proof. $2^n (x - 1/2)^n$ shows that $|a_0| \geqq 2^{n/2}$ is best possible.

Schur in his proof makes use of the fact that the discriminant of $f_n(x)$ has to be an integer. If we make use of this fact we easily obtain that, for large $n$, $|a_0| > (2^{1/2} + c)^n$. On p. 390 of his paper Schur constructs a polynomial of degree $2n$ with

$$a_0^{(2n)} = \frac{1}{2(2)^{1/2}} ((1 + 2^{1/2})^{n+1} - (1 - 2^{1/2})^{n+1}).$$

[4] Math. Zeit. vol. 1 (1918) pp. 377–402, see p. 389–391.

This seems the greatest possible value of $|a_0^{(2n)}|$.

In the same paper (Theorem XIII, pp. 397–398) Schur proves the following theorem: Let $a_0$ be a given integer, and let $f_n(z) = a_0z^n + \cdots + a_n$ be a polynomial with integer coefficients the roots of which either all have absolute value 1 and are different or all are in the interior of the unit circle (in which case multiple roots are permitted). Denote these roots by $z_1, z_2 \cdots, z_n$. Then

$$\limsup \frac{z_1 + z_2 + \cdots + z_n}{n} \leqq 1 - \frac{e^{1/2}}{2}. \tag{6}$$

Schur conjectures that the limit (6) is 0, and remarks that if $a_0 = 1$ this follows from a theorem of Kronecker, which asserts that in this case all the $z_i$ are roots of unity. We now prove Schur's conjecture.

THEOREM 4. *Let the $z_i$ be defined as above. Then*

$$\lim \frac{z_1 + z_2 + \cdots + z_n}{n} = 0.$$

First we can assume that $n$ tends to infinity (that is, for every $n$ there are only a finite number of equations satisfying the conditions of the theorem). Also if $f(z)$ has all its root in the interior of the unit circle then $z^2f(z) + z^nf(z^{-1}) = g(z)$ has all its roots on the unit circle and all are different. Also the sum of the roots of $f(z)$ and $g(z)$ are identical (p. 397). Thus it will suffice to consider polynomials having all their roots on the unit circle and distinct.

Therefore the discriminant of $D$ satisfies the inequality

$$1 \leqq D = a_0^{2n-2} \prod_{i<j\leqq n} (z_i - z_j)^2. \tag{7}$$

It follows from a result of Pólya (p. 395) that

$$|f(z_1, z_2, \cdots, z_n)| = \prod_{i<j\leqq n} (z_i - z_j) \leqq n^n. \tag{8}$$

Thus for at least one $z$

$$\prod_{j\neq i} |(z_i - z_j)| \leqq n. \tag{9}$$

To prove Theorem 4 it clearly suffices to show that the $z_i$ are uniformly distributed on the unit circle. Suppose this is not true. Then it follows from a result of Fekete[5] that there exists a $z_0$, $|z_0| = 1$, such that

[5] Ann. of Math. vol. 41 (1940) pp. 162–173, see pp. 165–166.

$$\prod_{j \neq i} |(z_0 - z_j)| > (1 + c_1)^n. \tag{10}$$

But then from (9) and (10)

$$|f(z_0, z_1, \cdots, z_{i-1}, z_{i+1} \cdots z_n) > \frac{(1 + c_1)^n}{n} f(z_1, z_2, \cdots, z_n).$$

If $z_1, z_2, \cdots, z_n$ are not uniformly distributed we can continue this process $c_2 n$ times, and thus obtain $y_1, y_2, \cdots, y_n$, $|y_i| = 1$, so that

$$|f(y_1, y_2, \cdots, y_n)| > \frac{(1 + c_1)^{c_2 n^2}}{n^{c_2 n}} |f(z_1, z_2, \cdots, z_n)|. \tag{11}$$

But from (7) and (11) we obtain

$$|f(y_1, y_2, \cdots, y_n)| > (1 + c_3)^{n^2} \frac{1}{a_0^{2n-2}} > n^n$$

which contradicts (8) and completes the proof of Theorem 4.

Szegö[6] proved the following theorem: Let $M$ be any closed set in the plane. Denote by $\omega_n(M, z_0)$ the maximum of $|f_n'(z_0)|$ for all polynomials $f_n(z)$ of degree $n$ which satisfy $|f_n(z)| \leqq 1$ for all $z$ in $M$. Assume that the transfinite diameter of $M$ is positive. Then

$$\lim \omega_n(M, z_0)^{1/n} < \infty.$$

Fekete[7] proved that if $z_0$ is not in $M$, $\lim \omega_n(M, z_0)^{1/n}$ exists and is finite if the transfinite diameter of $M$ is positive, and is infinite if the transfinite diameter of $M$ is 0.

Assume now that $z_0$ belongs to $M$. The following questions remain open: (1) Does $\lim \omega_n(M, z_0)^{1/n}$ exist? (2) Let the transfinite diameter of $M$ be 0. Is $\lim \omega_n(M, z_0)^{1/n} = \infty$?[7]

We are going to answer both questions in the negative. In fact we prove the following theorem.

THEOREM 5. *Let $M$ be the set consisting of* 0 *and* $1/2^k$, $k = 0, 1, 2, \cdots$. *Then*

$$\omega_n(M, 0) < c^n.$$

Clearly $M$ is closed and countable, thus its transfinite diameter is 0.

LEMMA. *Let $a$, $b$, $d$ be three real numbers, $d - b = b - a$. Then if $|f_n(z)| < 1$ for $a < z < b$,*

[6] Math. Zeit. vol. 23 (1925) pp. 45–61.
[7] Math. Zeit. vol. 26 (1927) pp. 324–344.

$$f_n'(d) < c_1^n/(b-a).$$

If $a=0$, $b=1$ the lemma follows from Szegö's[6] result. The general case follows by a linear transformation. As a matter of fact it is well known that $\omega_n(M, d)$ is given in this case by the Tchebychef polynomial belonging to $(a, b)$.

Now we prove Theorem 5. The equation $f_n^2(z)=1$ can have at most $2n$ real roots. Thus since $|f(1/2)^k)|<1$, $k=0, 1, \cdots$, we obtain that, for some $k>n+1$, $|f_n(z)|<1$ for all $1/2^k<z<1/2^{k+1}$. Thus by the lemma

$$|f_n'(0)| < 2^{n+1}c_1^n < c^n,$$

q.e.d.

THEOREM 6. *Let the set $M$ be defined as follows: $n_1<n_2<\cdots$ tend to infinity sufficiently fast. $M$ consists of the points $0$ and $1/2^u$ where $n_i\leqq u\leqq 2n_i+1$. Then* $\lim\, \omega_n(M, 0)^{1/n}$ *does not exist. In fact* $\limsup\, \omega_n(M, 0)^{1/n}=\infty$, $\liminf\, \omega_n(M, z_0)^{1/n}<\infty$.

As in Theorem 5 it follows that if $|f(1/2^u)|<1$ for $n_i\leqq u\leqq 2n_i+1$, $f(x)$ a polynomial of degree $n_i$, then $\omega_{n_i}(M, 0)<c^{n_i}$. Consider $f(x) =2^{n_i+1}\prod(x-1/2^k)$, $k\leqq 1, 2, \cdots, 2n_i+1$. The degree of $f(x)$ equals $2n_i+2$. Also $|f(x)|<1$ for all $x$ in $M$, and if $n_{i+1}$ tends to infinity sufficiently fast

$$(f'(0))^{1/2n_i+2} > (2^{n_i+1}/2^{(2n_i+1)^2})^{1/2n_i+2} \to \infty$$

q.e.d.

THEOREM 7. *Let $f_n(z)$ be a polynomial of degree $n$ with real coefficients. $|f_n(z)|<1$ for $-1\leqq z\leqq 1$. Then if $|z_0|\geqq 1$*

$$|f_n(z_0)| \leqq |T_n(z_0)|.$$

*Equality holds only for $f_n(z)=\pm T(z)$.*

In case $z_0$ is real this result is well known.

We are going to prove the following more general result: Let $|f_n(x_i)|\leqq 1$ where $x_1=-1$, $x_i$, $i=1, 2, \cdots, n-1$, are the roots of $T_n'(x)$ and $x_n=1$. Then for $|z_0|\geqq 1$

$$|f_n(z_0)| \leqq |T_n(z_0)|. \tag{12}$$

We have $(l_i(x)=\omega(x)/\omega'(x_i)(x-x_i)$, $\omega(x)=(1-x^2)T_n'(x))$

$$f_n(z_0) = \sum_{i=0}^{n} y_i l_i(x_0), \qquad |y_i| \leqq 1,\ y_i \text{ real.}$$

We evidently have for complex numbers $A$ and $B$, max $(|A+B|, |A-B|>A$. Thus $|f_n(z)|$ will be maximal if $y_i=\pm 1$. A simple geometric argument shows that the angle between any two of the vectors $(-1)^i l_i(z_0)$ is less than $\pi/2$ (since the interval $(-1, +1)$ subtends from $z_0$ at an angle not greater than $\pi/2$). But then clearly $|f_n(z_0)|$ is maximal if

$$f_n(z_0) = \pm \sum_{i=0}^{n} (-1)^i l_i(z_0) = \pm T_n(z_0).$$

Equality clearly occurs only if $f(z)=\pm T(z)$.

COROLLARY. *Let* $|f_n(z)| \leqq 1$ *for* $-1 \leqq z \leqq 1$, *also let* $f_n(z)$ *have real coefficients. Then for* $|z| \leqq 1$, $|f_n(z)| < |T_n(i)|$.

If we do not assume that the coefficients of $f_n(z)$ are real it is easy to give examples which show that $|f_n(z)|$ does not have to be less than $|T_n(i)|$. Trivially $|f_n(z)| \leqq \sum_{k=0}^{n} |l_k(i)|$. But in general max $|f_n(z)| < \sum_{k=0}^{n} |l_k(i)|$. I can not at present determine max $|f_n(z)|$ for $|z| \leqq 1$.

In the same way we can prove that if $f(z)=a_0z^n+ \cdots +a_n$ has real coefficients and $|f(z)| \leqq 1$ for $-1 \leqq z \leqq 1$ then $\sum_{k=0}^{n} |a_k|$ is maximal for $f(z)=\pm T_n(z)$. Szegö[8] proved the following stronger result: $|a_{2k}|+|a_{2k+1}|$ is maximal for $f(z)=\pm T_n(z)$.

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[8] Oral communication.