ON THE INTERIOR OF THE CONVEX HULL OF A EUCLIDEAN SET

WILLIAM GUSTIN

In this note we shall prove for each positive integer n the following theorem Δ_n concerning convex sets in an n-dimensional euclidean space.

THEOREM Δ_n. *Any point interior to the convex hull of a set E in an n-dimensional euclidean space is interior to the convex hull of some subset of E containing at most $2n$ points.*

This theorem is similar to the well known result that any point in the convex hull of a set E in an n-dimensional euclidean space lies in the convex hull of some subset of E containing at most $n+1$ points [1, 2].[1] In these theorems the set E is an arbitrary set in the space. The convex hull of E, denoted by $H(E)$, is the set product of all convex sets in the space which contain E.

A euclidean subspace of dimension $n-1$ in an n-dimensional euclidean space will be called a plane. Every plane in an n-dimensional euclidean space separates its complement in the space into two convex open sets, called open half-spaces, whose closures are convex closed sets, called closed half-spaces. If each of the two open half-spaces bounded by a plane L intersects a given set E, then L is said to be a separating plane of E; otherwise L is said to be a nonseparating plane of E.

In order to prove our sequence of theorems we shall make use of the following result: A point i is interior to the convex hull of a set E in an n-dimensional euclidean space if and only if every plane through i is a separating plane of E [1].

We prove our sequence of theorems by induction. The proof of Theorem Δ_1 is trivial and will be omitted. Now suppose that Theorem Δ_{n-1} is true for an integer $n>1$. We shall show that Theorem Δ_n is also true. To this end let i be a point interior to the convex hull of a set E in an n-dimensional euclidean space. We are to demonstrate that i is interior to the convex hull of some subset P of E containing at most $2n$ points.

First we show that i is interior to the convex hull of some finite subset Q of E. Since i is interior to $H(E)$, it is interior to a simplex

Received by the editors September 19, 1946, and, in revised form, November 16, 1946.

[1] Numbers in brackets refer to the references cited at the end of the paper.

lying in $H(E)$. Consider the $n+1$ vertices q_k $(k=1, \cdots, n+1)$ of such a simplex. The vertex q_k lies in $H(E)$ and hence, according to the previously mentioned result of Carathéodory and Steinitz, lies in the convex hull of some subset Q_k of E containing at most $n+1$ points. The set $Q=\sum_k Q_k$ is then a finite subset of E containing at most $(n+1)^2$ points. Evidently the convex hull of this set contains the simplex with vertices q_k and hence contains the point i in its interior.[2]

Since Q is finite, there exists a subset P of Q which contains the point i in the interior of its convex hull and which is irreducible with respect to this property. Let p be a definite point of P. Then i is not an interior point of $H(P-p)$, so some plane L through i is a non-separating plane of $P-p$. Let D be that one of the two open half-spaces bounded by L which is disjoint with $P-p$ and let D' be the other open half-space. Thus $P-p$ lies in the closed half-space $\overline{D}'$ complementary to D.

Since i is an interior point of the convex hull of P, the open half-space D contains a point of P. This point must be p, for D contains no point of $P-p$. Similarly the open half-space D' contains a point p' of P. We shall use this point p' later in the proof.

Consider an arbitrary point x of the closed half-space $\overline{D}'$. Since p lies in the complementary open half-space D, the line segment $H(p+x)$ intersects the boundary L of D in exactly one point which we denote by $\phi(x)$. Thus $\phi(x)$ is the projection of x from p onto L.

The projection ϕ is 1-1 over the subset $P-p$ of the closed half-space $\overline{D}'$. For suppose, to the contrary, that some two points p_1 and p_2 of $P-p$ project into the same point of L. The three points p, p_1, and p_2 are then collinear. Now p does not lie between the other two points, else the open half-space D containing p would contain at least one of these other two points. We may then assume p_1 and p_2 to be so labeled that a linear order of the three points is p, p_1, p_2. Therefore

$$p_1 \subset H(p + p_2) \subset H(P - p_1),$$

so the sets $H(P-p_1)$ and $H(P)$ are identical. The point i is then interior to $H(P-p_1)$ in contradiction to the irreducibility of P.

The projection of the convex hull of a set is the convex hull of the projection of that set, and the projection of an interior point of a convex set is an interior point of the projection of that set [3]. Therefore the point $\phi(i)=i$ is an interior point of the set $\phi(H(P-p))$ $=H(\phi(P-p))$ in the euclidean subspace L of dimension $n-1$. Ac-

[2] That i is interior to the convex hull of some finite subset of E may also be proved by the Heine-Borel theorem. I am indebted to the referee for the above proof.

cording to Theorem Δ_{n-1} the point i is an interior point in L of the convex hull of some subset P_L of $\phi(P-p)$ containing at most $2n-2$ points. Define

$$P^* = p + P\phi^{-1}(P_L) + p'.$$

Since the projection ϕ is 1-1 over $P-p$, the set $P\phi^{-1}(P_L)$ is a subset of P containing at most $2n-2$ points. Therefore P^* is a subset of P containing at most $2n$ points.

We shall show that i is interior to $H(P^*)$. First we notice that the coplanar set P_L lies in $H(P^*)$. For, if x is an arbitrary point of $P\phi^{-1}(P_L)$, then

$$\phi(x) \subset H(p + x) \subset H(P^*),$$

since both p and x lie in $H(P^*)$. Now consider the pyramid $H(p+P_L)$ whose apex p lies in D and whose base $H(P_L)$ lies in L. The point i is an interior point in L of the base $H(P_L)$ of this pyramid, so some closed hemisphere A with center i and base on L lies in $H(p+P_L)$. Similarly, some closed hemisphere A' with center i and base on L lies in the pyramid $H(p'+P_L)$. Evidently there exists a sphere I with center i such that $I \subset A + A' \subset H(p+P_L) + H(p'+P_L) \subset H(P^*)$. The point i is then interior to the convex hull of the subset P^* of P. From the irreducibility of P it follows that $P^* = P$. Therefore P contains at most $2n$ points.

Thus for every integer $n > 1$, Theorem Δ_{n-1} implies Theorem Δ_n. Since Theorem Δ_1 is true, we conclude by induction that Theorem Δ_n is true for each positive integer n.

The following example shows that the number $2n$ in Theorem Δ_n cannot be improved. Let i be the zero point of an n-dimensional vector space. Choose any n linearly independent and hence nonzero points in this space. Let E be the set consisting of these points and their vector negatives; E then contains $2n$ points. It is easy to show that the zero point i is interior to the convex hull of E but is not interior to the convex hull of any proper subset of E.

References

1. C. Carathéodory, *Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen*, Rend. Circ. Mat. Palermo vol. 32 (1911) §9.

2. E. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme*, J. Reine Angew. Math. vol. 143 (1913) §10.

3. ———, *Bedingt konvergente Reihen und konvexe Systeme*, J. Reine Angew. Math. vol. 146 (1916) §26.

University of California at Los Angeles