THEOREM 3. *If the conditions of Theorem* 2 *part* A *are satisfied and if in addition the quantities* ψ_1, ψ_2 *and* δ *satisfy the inequality*

$$2\pi > \delta(\csc \psi_1 - \csc \psi_2)$$

then the circle of convergence is not a cut for the function.

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A NOTE ON THE HILBERT TRANSFORM

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The Hilbert transform of $f(t)$, $-\infty < t < \infty$, is $1/\pi$ times the Cauchy principal value

$$\bar{f}(x) = P\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt = \lim_{\delta\to 0+}\int_{\delta}^{\infty} \frac{f(x+t)-f(x-t)}{t}\,dt.$$

If $f(t)\in L_p$, $p>1$, then $\bar{f}(x)\in L_p$, and a considerable literature is devoted to studying the relationship of such pairs of "conjugate" functions to the theory of functions analytic in a half-plane. More to the point of the present note is a series of papers studying the Hilbert transform along strictly real variable lines ([2, 3]; further bibliography in [2]).[1]

Much less is known about $\bar{f}(x)$ when $f(t)\in L_1$. Plessner found by applying complex variable methods to the theory of Fourier series that if $f(t)\in L_1$ then $\bar{f}(x)$ exists almost everywhere (see [1, p. 145]). Besicovitch [4] proved Plessner's result using only the theory of sets, starting from his own previous real variable investigation of the L_2 transform case. S. Pollard [5] showed how Besicovitch's proof could be extended to prove the existence a.e. of the principal value of the Stieltjes integral

$$\bar{f}(x) = P\int_{-\infty}^{\infty} \frac{dF(t)}{t-x},$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

where $F(t)$ is continuous and of bounded variation over $(-\infty, \infty)$. In general $\bar{f}(x)$ is not summable, but Kolmogoroff [6] found, using a contradiction argument, that there exists a constant A such that the set where $\bar{f}(x) > M > 0$ has measure at most $A\|f\|/M$, where $\|f\| = \int_{-\infty}^{\infty} |f(t)| dt$. Titchmarsh [7] was able to refine Besicovitch's existence proof so that it implied this bound, with a numerical value for A.

The present note contains a new direct real-variable proof of the Plessner existence theorem and the Kolmogoroff bound. In fact, this bound in a sense is the central tool for the existence proof, a device which allows for the first time the L_1 results to be obtained without recourse to the L_2 transform theory.

LEMMA 1. *If* $c_i > 0$ *and*

$$g(x) = \sum_{i=1}^{n} \frac{c_i}{x - a_i},$$

then the set of points where $g(x) > M$ $(M > 0)$ *consists of n intervals whose total length is precisely* $(\sum c_i)/M$. *The set where* $g(x) < -M$ *has the same length.*

Since $g(a_i-) = -\infty$, $g(a_i+) = \infty$ and $g'(x) < 0$ for all x, there are precisely n points m_i such that $g(m_i) = M$, and $a_i < m_i < a_{i+1}$, $i = 1, \cdots, n-1$, $a_n < m_n$. The set where $g(x) > M$ thus consists of the intervals (a_i, m_i) and has the total length

$$\sum_{i=1}^{n} (m_i - a_i) = \sum_{i=1}^{n} m_i - \sum_{i=1}^{n} a_i. \tag{1}$$

But the numbers m_i are the roots of the equation

$$\sum_{i=1}^{n} \frac{c_i}{x - a_i} = M,$$

whose cross-multiplied form is

$$\sum_{i=1}^{n} c_i \left[\prod_{j \neq i} (x - a_j) \right] = M \prod_{i=1}^{n} (x - a_i),$$

or

$$Mx^n - [M \sum a_i + \sum c_i] x^{n-1} + \cdots = 0,$$

so that

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} a_i + \frac{1}{M} \sum_{i=1}^{n} c_i. \tag{2}$$

The first part of the lemma follows from (1) and (2); the proof for $g(x) < -M$ is almost identical.

LEMMA 2. *Let $F(t)$ be increasing over $(-\infty, \infty)$ with finite total variation $V(F)$. If $(x_j-\delta_j, x_j+\delta_j)$, $j=1, \cdots, n$, are disjoint intervals such that*

$$\int_{-\infty}^{x_j-\delta_j} + \int_{x_j+\delta_j}^{\infty} \frac{dF(t)}{t-x_j} > M > 0, \tag{3}$$

then $\sum \delta_j \leqq 4V(F)/M$. The same inequality is implied if the integral is less than $-M$, $j=1, \cdots, n$.

Let t_i, $i=1, \cdots, N$ be a finite subdivision including the points $x_j-\delta_j$, x_j, $x_j+\delta_j$ for $j=1, \cdots, n$, and such that the approximating Riemann sums for (3), with the integrand evaluated at the left-hand end points, remain greater than M. Thus, if $\Delta_i = F(t_{i+1}) - F(t_i)$,

$$\sum_{i \notin I_j} \frac{\Delta_i}{t_i - y} > M \tag{4}$$

for $y = x_j$, where the set I_j of omitted indices is defined by

$$\bigcup_{i \in I_j} (t_i, t_{i+1}) = (x_j - \delta_j, x_j + \delta_j).$$

Since the left member of (4) is an increasing function of y for $x_j-\delta_j < y < x_j+\delta_j$, the inequality (4) holds for $x_j \leqq y < x_j+\delta_j$. For every such y one of the following inequalities is therefore satisfied:

$$\sum_{i=1}^{N-1} \frac{\Delta_i}{t_i - y} > \frac{M}{2}, \qquad \sum_{i \in I_j} \frac{\Delta_i}{t_i - y} < -\frac{M}{2}.$$

Applying Lemma 1 and summing over j, we have

$$\sum \delta_j \leqq \sum_{i=1}^{N-1} \frac{2\Delta_i}{M} + \sum_{j=1}^{n} \sum_{i \in I_j} \frac{2\Delta_i}{M} \leqq \frac{4}{M} \sum_{i=1}^{N-1} \Delta_i \leqq \frac{4}{M} V(F).$$

To prove the second part of the lemma we only need to observe that the integral in (3) is less than $-M$ if and only if after replacing $F(t)$ by $-F(-t)$ and x_j by $-x_j$ it is greater than M.

COROLLARY. *If $F(t)$ is of bounded variation in Lemma 2 then $\sum \delta_j \leqq 8V(F)/M$.*

This follows at once upon applying the lemma to the increasing and decreasing parts, F_1 and F_2, of F, using $V(F) = V(F_1) + V(F_2)$.

Preliminary to the theorem we remark that if $f(t)$ has the value 1 in

(a, b) and 0 elsewhere, then its Hilbert transform exists except at the two points a and b, and has the value $\log |(x-b)/(x-a)|$. In particular the Hilbert transform of any step function exists except at a finite number of points.

THEOREM. *Let $F(t)$ be of bounded variation over $(-\infty, \infty)$. Then its Hilbert-Stieltjes transform*

$$\bar{f}(x) = P\int_{-\infty}^{\infty} \frac{dF(t)}{t-x}$$

exists almost everywhere, and, for every positive M, the set where $\bar{f}(x) > M$ has measure at most $16V(F)/M$, as does the set where $\bar{f}(x) < M$.

We first prove the existence of $\bar{f}(x)$. It is sufficient to show that, given ϵ, for every x except in a set of measure less than ϵ

$$\left| \int_{x-\delta}^{x-\delta'} + \int_{x+\delta'}^{x+\delta} \frac{dF(t)}{t-x} \right| \leqq \epsilon \tag{5}$$

for all sufficiently small δ and δ'. Now the absolutely continuous part of F can be approximated to within ϵ' by the integral F_1 of a step function h, $F_1(t) = \int_{-\infty}^{t} h(t)dt$, and the singular part of F can be approximated to within ϵ' by a singular function F_2 whose variation is confined to a closed set of measure 0, that is, which is constant on the intervals of an open set M whose complement has measure zero. Thus, taking $\epsilon' = \epsilon^2/192$, we have $F = F_1 + F_2 + F_3$, where $V(F_3) < \epsilon^2/96$. Let E_ϵ be the set of x for which the inequality

$$\left| \int_{x-\delta}^{x-\delta'} + \int_{x+\delta'}^{x+\delta} \frac{dF_3(t)}{t-x} \right| \leqq \frac{\epsilon}{3} \tag{6}$$

fails to hold for arbitrarily small δ and δ' $(\delta' < \delta)$. Then for every x in E_ϵ

$$\left| \int_{-\infty}^{x-\Delta} + \int_{x+\Delta}^{\infty} \frac{dF_3(t)}{t-x} \right| > \frac{\epsilon}{6} \tag{7}$$

for arbitrarily small Δ. By Vitali's theorem a disjoint sequence of intervals $(x_i - \Delta_i, x_i + \Delta_i)$ satisfying (7) can be chosen so as to cover E_ϵ except for a set of measure 0. Then by Lemma 2, corollary, $m(E_\epsilon) \leqq 2\sum \Delta_i \leqq 2 \cdot 8V(F_3) \cdot 6/\epsilon < \epsilon$.

Since $F_1(t)$ is the integral of a step function, its Hilbert-Stieltjes transform (the Hilbert transform of $h(t)$) exists except at a finite number of points, which we add to E_ϵ. Since $F_2(t)$ is constant on the intervals of M, its Hilbert-Stieltjes transform obviously exists except on the complement of M, which we add to E_ϵ. Thus if x is not in the enlarged E_ϵ there is a Δ such that, for all δ and δ' less than Δ, (6) holds for F_1, F_2 and F_3, and hence (5) holds, as was to be proved.

The second part of the theorem follows immediately from Lemma 2, corollary, where the intervals $(x_i-\delta_i,\ x_i+\delta_i)$ are chosen by Vitali's theorem to cover almost entirely the set where $\bar{f}(x)>M$ $(<-M)$, so that the measure of this set is not greater than $2\sum\delta_i\leqq 16\,V(F)/M$.

COROLLARY. *If* $0<p<1$ *and* $p+q>1$, *then* $|\bar{f}(x)|^p/(1+|x|)^q\in L_1$.

This follows immediately from the fact that the decreasing function on $(0, \infty)$ which is equimeasurable with $|\bar{f}(x)|$ is dominated by K/x.

In case $F(t)$ is singular and increasing, it can be shown with little difficulty that the constant 16 can be replaced by 1, and this is best possible since $1/x$ itself is the Hilbert-Stieltjes transform of the function $F(t)$ which is 1 when $t<0$ and 0 when $t\geqq 0$. This is probably the correct value of the constant in the general case.

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