NOTE ON ALMOST-ALGEBRAIC NUMBERS

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1. **Introduction.** According to a theorem of J. Liouville,[2] if θ is an algebraic number of degree n, then any approximation by rationals, p/q, is of such a nature that

$$(1) \qquad |\theta - p/q| \geqq kq^{-n}$$

for a positive constant k. Liouville constructed his transcendental numbers as the limit of special sequences of rationals, p/q, which violated condition (1) regardless of the values of k and n, as $q \to \infty$. Thus Liouville constructed *almost-rational* numbers.

E. Maillet[3] likewise found a lower bound for $\theta - \alpha$ where now θ is approximated by the quadratic numbers, α. He then violated his lower bound by substituting for θ the value of an almost periodic simple continued fraction and for α a quadratic number, namely a periodic simple continued fraction that θ almost represented. Thus he constructed an *almost-quadratic* transcendental.

It is an elementary matter to find a lower bound for $\theta - \alpha$, where we now approximate θ by an algebraic number not necessarily rational or quadratic. We could then try several departures. We could, for example, try to construct almost-cubic or almost-biquadratic transcendentals.[4] On the other hand, we could use a *diagonal* method, that is, we could consider the limit of a rapidly converging sequence of algebraic numbers whose degree becomes indefinite. For example, a root of a power series with rational coefficients is the limit of a sequence of (algebraic) roots of the partial sums, and the speed of convergence is regulated by the remainder. If the remainder is too small we find that the root of our power series can be approximated too closely by algebraic numbers of varying degrees, namely the roots of

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[2] J. Liouville, *Sur des classes très étendues des quantités dont la valeur n'est ni algébrique ni même réductible à des irrationelles algébriques*, J. Math. Pures Appl. vol. 16 (1851).

[3] E. Maillet, *Théorie des nombres transcendents et des propriétés arithmétiques des fonctions*, Paris, Gauthier-Villars, 1906, chap. 7.

[4] For instance, E. Maillet, op. cit., pp. 22, 100, considers certain "rapidly converging" power series with rational coefficients and algebraic values of the argument. The value of such a series is shown to be almost algebraic when regarded as the limit of the algebraic partial sums (which lie in the field generated by the argument and therefore are of no higher degree than the argument).

the partial sums. Thus the root of our power series must be transcendental. By this method we can obtain some transcendental numbers which seem to have previously escaped notice.

2. **Approximation of a fixed algebraic number by an arbitrary algebraic number.** We let θ be a *fixed* algebraic number of degree n whose denominator is t_0, that is, θ satisfies the equation with integral coefficients

$$t_0\theta^n + t_1\theta^{n-1} + \cdots + t_n = 0.$$

We consider α the root of an *arbitrary* polynomial with integral coefficients

$$\phi(x) = f_0x^m + f_1x^{m-1} + \cdots + f_m.$$

Denoting $\max|f_i|$ by $A[\phi(x)]$, we shall measure the closeness of α to θ by the smallness of $\phi(\theta)$ in terms of $A[\phi(x)]$.

Now setting $\bar{\phi}(x) = f_0x^m + f_1t_0x^{m-1} + \cdots + f_0t_0^m$, we find $\phi(\theta)$ can be written as $\bar{\phi}(t_0\theta)/t_0^m$, where $t_0\theta$ is an algebraic integer. Thus if we multiply $\phi(\theta)$ by its conjugates, we obtain

$$\prod \phi(\theta) = \left\{ \prod \bar{\phi}(t_0\theta) \right\} / t_0^{nm} = M[\phi(\theta)]/t_0^{nm}, \tag{2}$$

where $M[\phi(\theta)]$ is an integer. If $\phi(\theta) \neq 0$, then the same inequality holds for the conjugates of θ and thus

$$| M[\phi(\theta)] | \geqq 1. \tag{3}$$

For any conjugate of θ, we find easily

$$| \phi(\theta') | \leqq A[\phi(x)]T_1^m, \tag{4}$$

where T_1 is a constant determined by θ.

Hence if $\phi(\theta) \neq 0$, we obtain from (2), (3), and (4)

$$| \phi(\theta) | \geqq 1/\left\{ (A[\phi(x)])^{n-1} T_1^{m(n-1)} t_0^{mn} \right\}.$$

In short, *if θ is algebraic of degree n, then for an arbitrary polynomial $\phi(x)$ of degree m with integral coefficients no greater than $A[\phi(x)]$ in absolute value,*

$$| \phi(\theta) | \begin{cases} = 0 \text{ or} \\ \geqq 1/\left\{ (A[\phi(x)])^{n-1} T_2^m \right\}, \end{cases} \tag{5}$$

where T_2 depends only on θ. We should note in passing that if $\phi(x) = -p + qx$, then (5) is Liouville's theorem.

3. **Construction of transcendentals.** We shall construct a power series $\sigma(x) = a_0x^{e_0} + a_1x^{e_1} + a_2x^{e_2} + \cdots$ $(e_i \geqq 0)$, where a_i are rational fractions r_i/s_i $(\neq 0)$, and e_i are increasing integral exponents. For sim-

plicity set $a_0=1$. Let $\sigma_h(x)$ be the hth partial sum (up to and including the exponent e_h). Then of the coefficients a_i in $\sigma_h(x)$ let g_h be the maximum $|a_i|$ $(g_h \geqq 1)$ and let d_h be the least common multiple of the denominators s_i in $\sigma_h(x)$. Then if as $h \to \infty$, e_{h+1} is of a higher order of infinity[5] than e_h, $\log g_h$, and $\log d_h$, we find that $\sigma(x)$ *is transcendental for all algebraic* x *within its circle of convergence*, except $x=0$.

To see this, let x take the algebraic value θ of degree n within a circle of convergence of radius ρ. We suppose $E(\sigma)$ to be an arbitrary polynomial of degree k with integral coefficients of which the greatest has the absolute value $A[E(\sigma)]$. Then we wish to show $E(\sigma(\theta)) \neq 0$.

First we write $E(\sigma(x))$ as partial sum and remainder

$$E(\sigma(x)) = E(\sigma_h(x)) + R_h(x).$$

Now $\phi_h^k(x) = d_h E(\sigma_h(x))$ is a polynomial of degree ke_h with integral coefficients. Its maximum coefficient $A[\phi_h(x)]$ easily satisfies the condition

$$\begin{aligned} A[\phi_h(x)] &\leqq d_h^k \sum_{\kappa=0}^{k} A[E(\sigma)](h+1)^{\kappa} g_h^{\kappa} \\ &\leqq d_h^k (k+1) A[E(\sigma)](h+1)^k g_h^k \end{aligned}$$

since any coefficient in the polynomial $\phi_h(x)$ is less than the value this polynomial assumes when we replace each a_i by g_h and set $x=1$.

Furthermore, since $E(\sigma)$ has only a limited number of roots, $E(\sigma_h(\theta))$ can not vanish for all h. For, when h changes to $h+1$, $\sigma_h(\theta)$ changes by $a_{h+1}\theta^{e_{h+1}}$, which approaches zero as h approaches infinity, although this term never vanishes, $\theta=0$ having been excluded. Thus by (5), for an infinity of values of h,

$$|d_h^k E(\sigma_h(\theta))| \geqq \{d_h^k (k+1) A[E(\sigma)](h+1)^k g_h^k\}^{-n+1} T_2^{-ke_h}.$$

But since h can not exceed 2^{e_h} (or even e_h), we can simplify the last inequality and obtain

$$|E(\sigma_h(\theta))| \geqq d_h^{-nk} g_h^{-(n-1)k} T_3^{-e_h}, \tag{6}$$

where T_3 depends only on θ and on $E(\sigma)$ as a polynomial in σ.

On the other hand $R_h(\theta) = E(\sigma(\theta)) - E(\sigma_h(\theta))$ and, by elementary considerations,

$$|R_h(\theta)| \leqq T_4 |\sigma(\theta) - \sigma_h(\theta)| \leqq T_5 |(\theta/\rho)^{e_{h+1}}/(1-\theta/\rho)| \tag{7}$$

where T_5 is determined by $\sigma(x)$ and $E(\sigma)$, independently of h.

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[5] If we have two quantities u and v each depending on h we say u is of a higher order of infinity than v, if $u/v \to \infty$ with h.

By our condition on the order of magnitude of e_{h+1}, it follows from (6) and (7) that for some h, $|E(\sigma(\theta))| \geqq |E(\sigma_h(\theta))| - |R_h(\theta)| > 0$; and $\sigma(\theta)$ can not satisfy the algebraic equation $E(\sigma) = 0$. Q.E.D.

A simple method of satisfying the conditions on $\sigma(x)$ is to take $\sigma_1(x) = 1 + x^{1!} + x^{2!} + \cdots + x^{n!} + \cdots$, $|x| < 1$. Then $\sigma_1(x)$ and x are not both algebraic unless $x = 0$, and indeed, the values of x $(\neq 0)$ for which $\sigma_1(x)$ is algebraic are some instances of the type of transcendentals under discussion.

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