A REMARK ON LOCALLY COMPACT ABELIAN GROUPS

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It has recently been shown by Halmos [1][1] that there exists a compact topological group which is algebraically isomorphic to the additive group of the real line, an example being given by the character group of the discrete additive group of the rationals. Exploiting his argument a bit further it is easy to see that the most general such example is the direct sum of $\aleph$ replicas of the one already given where $\aleph$ is a cardinal such that $2^{\aleph} \leqq C$. This having been observed it naturally occurs to one to ask for the most general locally compact topological group with the algebraic structure in question. It is the purpose of the present note to give a complete answer to this question. We shall do so by giving a proof of the following theorem.

THEOREM. *Let G be a locally compact topological group which is algebraically isomorphic to the additive group of a linear space over the rationals. Then G is isomorphic*[2] *to a direct sum of four groups G_1, G_2, G_3, and G_4 which may be described as follows. G_1 is the additive group of an n-dimensional* ($n=0, 1, 2, \cdots$) *real linear space with the customary Euclidean topology; that is, an n-dimensional vector group. G_2 is the direct sum of $\aleph$ replicas of the character group of the discrete additive group of the rationals. G_3 is a discrete group algebraically isomorphic to an $\aleph^-$-dimensional linear space over the rationals. G_4 is algebraically isomorphic to the additive group of a linear space over the rationals and contains a compact subgroup H whose quotient group is a discrete torsion group. H is a direct sum of finitely or infinitely many groups each of which is isomorphic to the additive topological group of the p-adic integers for some prime p. The groups G_1, G_2, G_3, and G_4 are unique up to an isomorphism. Furthermore* (*modulo isomorphisms*) *the groups G_4 and H determine one another. Thus the numbers n, $\aleph$, $\aleph^-$, and the function f from the primes to the cardinal numbers which gives the multiplicity of occurrence of each p-adic group in H form a complete set of invariants for the topological group G. There exists a G having any assigned set of invariants. Finally G is algebraically isomorphic to the additive group of the real numbers if and only if it has continuum many elements; that is, if and only if n, $\aleph$, $\aleph^-$ and f are chosen so that no G_i has more*

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[1] Numbers in brackers refer to the bibliography at the end of the paper.

[2] In this paper the word isomorphic without qualification means isomorphic as a topological group.

than continuum many elements and at least one G_i has exactly continuum many elements.

We begin the proof by observing as in [1] that an Abelian group J is algebraically isomorphic to a linear space over the rationals if and only if it is torsion free in the sense that the only element of finite order is zero and divisible in the sense that $nx=y$ is solvable for x in J for each y in J and each $n=1, 2, \cdots$; and that furthermore if J is provided with a discrete or compact topology under which it is a topological group then it is divisible (resp. torsion free) if and only if its character group[3] is torsion free (resp. divisible).[4]

In this and the following two paragraphs we subtract out the G_i one by one. First of all it follows at once from the structure theorem on page 110 of [3] that G is isomorphic to the direct sum of an n-dimensional vector group G_1 and a locally compact group J_1 where J_1 admits a compact subgroup J_2 with a discrete quotient group. Clearly J_1 is Abelian, divisible and torsion free.

Let G_2 be the component of zero in J_1. Since J_1/J_2 is discrete it follows that $G_2 \subseteq J_2$ and hence that G_2 is compact. We shall show first that G_2 is a direct summand of J_2 and then from this that it is a direct summand of J_1. Since G_2 is connected its character group is torsion free so that it itself is divisible. It follows at once from this and the fact that J_2 is torsion free that J_2/G_2 is torsion free. Let G_2' be the set of all elements of the character group C of J_2 which annihilate G_2. Then since G_2' is isomorphic to the character group of J_2/G_2 it follows that G_2' is divisible. Consider the family of all subgroups J of C such that $J \cap G_2' = 0$. It is a consequence of a well known lemma of Zorn that there exists a maximal such subgroup J_3 and it follows easily from the divisibility of G_2' that $G_2' + J_3 = C$. Since C is discrete it is the direct sum of J_3 and G_2'. Hence J_2 must be the direct sum of G_2 and the annihilator J_3' of J_3 in J_2. By applying the argument just used in obtaining J_3 to the set of all (not necessarily closed) subgroups of J_1 which contain J_3' and intersect G_2 in zero we can conclude the existence of a subgroup J_4 of J_1 which contains J_3' and is such that $G_2 \cap J_4 = 0$ and $G_2 + J_4 = J_1$. We shall show that J_1 is the direct sum of G_2 and J_4. It will follow in particular that J_4 is closed. Let O be an arbitrary open subset of J_1 which contains

[3] The reader is referred to [2] or [3] for a discussion of the theory of characters. We shall make free and frequent use of the results of this theory in the course of this paper.

[4] The example indicated in the last paragraph of this paper shows that this relationship between divisibility and torsion freeness does not hold for general locally compact groups in spite of the assertion to the contrary in [1].

zero. Let $O' = O \cap J_2$. Since J_2 is the direct sum of G_2 and J_3' there exists an open subset O'' of J_2 which contains zero and is such that if $x+y \in O''$ where $x \in G_2$ and $y \in J_3'$ then $x \in O'$ and $y \in O'$. Now since J_1/J_2 is discrete J_2 is an open subset of J_1. It follows that O'' is an open subset of J_1. Suppose that $x+y \in O''$ where $x \in G_2$ and $y \in J_4$. Then since $x+y \in J_2$ we have $y \in J_3'$ and hence $x \in O' \subseteq O$ and $y \in O' \subseteq O$. Thus J_1 is indeed the direct sum of G_2 and J_4. That G_2 is of the character indicated in the statement of the theorem follows at once from the fact that its character group is divisible and torsion free and hence isomorphic to the additive group of a discrete linear space over the rationals.

We now decompose J_4. By the structure theorem for Abelian locally compact groups referred to above J_4 contains a compact subgroup H with a discrete quotient group. (As a matter of fact J_3' is such a subgroup.) Clearly J_4 is a linear space over the rationals. Let G_4 be the smallest linear subspace containing H and let G_3 be a linear subspace such that $G_3 \cap G_4 = 0$ and $G_3 + G_4 = J_4$. That G_3 exists is an easy consequence of Zorn's lemma. The proof that J_4 is the direct sum of G_3 and G_4 is analogous to but easier than the proof that J_1 is the direct sum of G_2 and J_4 and will be left to the reader. G_3 obviously has the character required in the statement of the theorem and so has G_4 insofar as its relationship to H is concerned. To show that H has the required properties we observe that it is torsion free and totally disconnected as well as compact. Thus its character group is a discrete divisible torsion group. Now countable discrete Abelian torsion groups have been completely analyzed and the part of that analysis applying to divisible groups applies without essential change in the noncountable case (see for example [4]) and assures us that the character group of H is a weak[5] direct sum of a finite or infinite family of groups each of which is of the form R_p/N where p is a prime, R_p is the subgroup of the additive group of the rationals consisting of those numbers whose denominator may be taken a power of p, and N is the group of integers. It is readily verified that R_p/N is isomorphic to the character group of the additive topological group of the p-adic integers and hence that H has the required form.

It remains to show that G_1, G_2, G_3, and G_4 are unique to within an isomorphism and that to within an isomorphism G_4 and H determine each other uniquely, the other statements of the theorem being obvious. Now the uniqueness of the G's follows at once from the fact that

[5] The weak direct sum of a family of discrete groups is the subset of the ordinary direct sum consisting of those elements with only a finite number of nonzero components and retopologized so as to be discrete.

they may all be defined in an invariant manner. In fact[6] G_1+G_2 is the component of zero in G and G_2+G_3 is the annihilator in G of the component of zero in the character group of G. G_2 then appears as the common part of G_1+G_2 and G_2+G_3 while G_4 is isomorphic to the quotient group of G modulo the sum of G_1+G_2 and G_2+G_3. Finally G_1 and G_3 are respectively isomorphic to the quotient groups of G_1+G_2 and G_2+G_3 modulo G_2.

We consider finally the relationship between H and G_4. First of all suppose that H_1 and H_2 are two compact subgroups of G_4 having discrete quotient groups. Then H_1+H_2 is also a compact subgroup with a discrete quotient group. Thus in order to show that H_1 and H_2 are isomorphic we may confine our attention to the case in which $H_1 \subseteq H_2$. Since H_2/H_1 is both compact and discrete it must be finite. Thus if C_1 and C_2 are the respective character groups of H_1 and H_2 then C_1 is isomorphic to the quotient of C_2 modulo a finite subgroup. But it is obvious that factorization modulo a finite subgroup cannot change the number of summands of the form R_p/N in C_2. Thus C_1 and C_2 and hence H_1 and H_2 are isomorphic.

Suppose conversely that G_4 and G_4^- are torsion free divisible locally compact Abelian groups containing isomorphic compact subgroups H and H^- such that G_4/H and G_4^-/H^- are discrete torsion groups. Given $x \in G_4$ let r be the least integer such that $rx \in H$. Let $y \in H^-$ correspond to rx in H under the given isomorphism between H and H^-. Since G_4^- is divisible and torsion free there is a unique z in G_4^- with $rz = y$. Let $\phi(x) = z$. It is readily verified that the function ϕ so defined sets up an isomorphism between G_4 and G_4^-. This completes the proof of the theorem.

We should probably call attention here to the general phenomenon of which the relationship between H and G_4 is a special case. If T is an arbitrary torsion free Abelian group then it is easy to show that T is imbeddable in an essentially unique Abelian group S which is torsion free, divisible and such that S/T is a torsion group. One need only repeat the argument used in constructing the rational numbers from the integers. Furthermore if T is equipped with a topology under which it is a topological group then S may be topologized in only one way so that the relative topology in T coincides with the given one, so that T is closed and so that S/T is discrete. This topology is defined by taking the neighborhoods of zero in T as the neighborhoods of zero in S. It is clear that if T is locally compact then so is S. Calling S the divisible extension of T we see that G_4 is simply the

[6] We are supposing here that the original G_i have been replaced by isomorphic subgroups of G as we of course always may.

divisible extension of H. It is not difficult to see that the additive topological group of the field of all p-adic numbers for a given p is the divisible extension of the additive topological group of p-adic integers for the same p. It follows from this last fact that whenever H has only a finite number of summands then G_4 is a direct sum of finitely many p-adic number groups.

In closing we remark that if H is the direct sum of $\aleph$ p-adic integer groups for the same prime p where $\aleph \geqq \aleph_0$ then the character group of G_4/H is the direct sum of $2^{\aleph}$ p-adic integer groups while that of H is the weak direct sum of only $\aleph$ groups of the form R_p/N. It follows at once that the character group of G_4 while torsion free cannot be divisible. On the other hand if $\aleph$ is finite then it is not difficult to see that G_4 and its character group are isomorphic. Furthermore an infinite number of summands for H does no harm provided that each prime appears only a finite number of times and one may show that G has a divisible character group if and only if $f(p)$ is finite for every prime p and is isomorphic to its character group if and only if $f(p)$ is finite for every prime p and $\aleph = \aleph^-$.

Bibliography

1. P. R. Halmos, *Comment on the real line*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 877–878.

2. L. Pontrjagin, *Topological groups*, Princeton, 1939.

3. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Scientifiques et Industrielles, no. 869, Paris, 1940.

4. L. Zippin, *Countable Abelian torsion groups*. Ann. of Math. vol. 36 (1935) pp. 86–99.

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