THE REPRESENTATION OF e^{-x^λ} AS A LAPLACE INTEGRAL

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According to a theorem of Bochner [1, p. 498][1] the function e^{-x^λ}, for any fixed value of λ in $0<\lambda<1$, is completely monotonic and admits a unique representation

$$e^{-x^\lambda} = \int_0^\infty e^{-xt} d\alpha_\lambda(t), \qquad 0 \leqq x < \infty,$$

where $\alpha_\lambda(t)$ is bounded and increasing. It follows further from a criterion of Hille and Tamarkin [2, p. 903] that the function also has the form

$$e^{-x^\lambda} = \int_0^\infty e^{-xt}\phi_\lambda(t)\, dt. \tag{1}$$

One can conclude therefore, since $\alpha_\lambda'(t) = \phi_\lambda(t)$, that $\phi_\lambda(t)$ is positive almost everywhere and that

$$\int_0^\infty \phi_\lambda(t) dt < \infty.$$

For this last integral is the total variation of $\alpha_\lambda(t)$, suitably normalized.

Further information concerning $\phi_\lambda(t)$ may be derived from some general results of Post. Let γ be the contour

$$\frac{x}{a} + \frac{|y|}{b} = 1$$

where a and b are fixed and positive; their precise values are a matter of indifference. The principal branch of e^{-z^λ} is holomorphic in the sector to the right of γ, and is moreover of zero type there since $0<\lambda<1$. If e^{-z^λ} is denoted by $f(z)$, the theory of Post [3, p. 730] shows that the limit

$$L[f; t] = \lim_{kh\to t, h\to 0+} \frac{(-1)^k}{k!} h^{-k-1} f^{(k)}\left(\frac{1}{h}\right) = \frac{1}{2\pi i}\int_\gamma e^{zt}e^{-z^\lambda} dz$$

exists for all $t>0$; γ must be traced so that the origin is at the left. But according to the Post-Widder inversion theorem [4, p. 288] $L[f; t]$ is the inverse Laplace transform of e^{-x^λ}, and so must be equal

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[1] Numbers in brackets refer to the references cited at the end of the paper.

to $\phi_\lambda(t)$ almost everywhere. Explicitly then

$$\phi_\lambda(t) = \frac{1}{2\pi i}\int_\gamma e^{zt}e^{-z^\lambda}dz \tag{2}$$

almost everywhere.

A further advantage of the approach we have used is this. Post has shown [3, p. 733] that the existence everywhere of $L[f; t]$ implies its continuity. It follows that by altering $\phi_\lambda(t)$ on a set of measure zero at most it becomes continuous for $t>0$. With this alteration (2) is valid for all $t>0$. The explicit evaluation of the contour integral in (2) is of considerable interest, and this is our next problem.

Certain special cases already appear in the literature in one form or another. For example, if $\lambda=1/2$ it is known [5, pp. 401–402] that

$$\phi_{1/2}(t) = \frac{1}{\pi}\int_0^\infty e^{-tu}\sin u^{1/2}du = \frac{1}{2\pi^{1/2}}\frac{1}{t^{3/2}}e^{-1/4t}. \tag{3}$$

Very recently Humbert [6] has discussed the problem of representing the derivative of e^{-x^λ} as a Laplace integral. His method, which is purely formal, leads to the following expression for $\phi_\lambda(t)$:

$$\phi_\lambda(t) = -\frac{1}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\sin \pi\lambda k\,\frac{\Gamma(\lambda k+1)}{t^{\lambda k+1}}\cdot \tag{4}$$

In particular, if $\lambda=2/3$

$$\phi_{2/3}(t) = -\frac{1}{2(3\pi)^{1/2}}\frac{1}{t}e^{-2/27t^2}W_{-1/2,-1/6}\left(-\frac{4}{27t^2}\right).$$

The present note provides a rigorous justification for the formula (4)

We shall prove first that (1) is inverted by

$$\phi_\lambda(t) = \frac{1}{\pi}\int_0^\infty e^{-tu}e^{-u^\lambda\cos\pi\lambda}\sin(u^\lambda\sin\pi\lambda)du, \tag{5}$$

the proof being the simple matter of demonstrating the equality of the right-hand members of (2) and (5). To this end consider a contour δ defined as follows. Let A and B be the intersections in the second and third quadrants respectively of the contour $\gamma=\gamma(a, b)$ with a circle ζ_1, center at the origin, radius $R>\max(a, b)$. Let ζ_2 be a circle of radius $\rho<ab(a^2+b^2)^{-1/2}$, center also at the origin. The intersections of ζ_1 and ζ_2 with the negative real axis will be denoted by C and D respectively. δ is then the contour obtained by starting at C, following the negative real axis to the right as far as D, circumnavigating the

origin once around the circle ζ_2, retracing the negative axis from D back to C, then tracing in order the arc CB of ζ_1, the portion of γ between B and the point $(a, 0)$, then between $(a, 0)$ and A, returning finally to C along the arc AC of ζ_1. Then for any $t>0$

$$\frac{1}{2\pi i}\int_{\delta} e^{zt}e^{-z^{\lambda}}dz = 0. \tag{6}$$

Now let $R\to\infty$, $\rho\to 0$. The parts of the integrals taken along the curved portions of δ vanish. What is left is the integral along γ, and along the negative real axis traced twice. The first of these is the right-hand member of (2). The second, taking into account the fact that $e^{-z^{\lambda}}$ changed branches when the origin was circumnavigated, is precisely the negative of the right-hand member of (5). In view of (6) the two right-hand members in question must therefore be equal. This completes the proof of (5).

To derive (4) observe that by (5)

$$\begin{aligned}\phi_{\lambda}(t) &= \frac{1}{\pi} I\left\{\int_0^{\infty} e^{-tu}e^{-(e^{-i\pi}u)^{\lambda}}du\right\}\\ &= \frac{1}{\pi} I\left\{\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}e^{-i\pi\lambda k}\frac{\Gamma(\lambda k+1)}{t^{\lambda k+1}}\right\},\end{aligned}$$

where "I" denotes "imaginary part of."

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