

## EQUIVALENCE IN A CLASS OF DIVISION ALGEBRAS OF ORDER 16

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Let  $\mathbb{C}$  be a Cayley-Dickson division algebra over an arbitrary field  $\mathfrak{F}$  with principal equation

$$(1) \quad x^2 - t(x)x + n(x) = 0$$

and involution

$$(2) \quad S: x \leftrightarrow xS = t(x) - x.$$

We are concerned with division algebras  $\mathfrak{A}$  of order 16 over  $\mathfrak{F}$  defined in the following way: let  $\mathbb{C}_o$  be a division algebra (of order 8) over  $\mathfrak{F}$  with the same elements as  $\mathbb{C}$  but with multiplication denoted by  $xoy$ ; further let  $\mathfrak{A} = \mathbb{C} + v\mathbb{C}$ , multiplication<sup>1</sup> in  $\mathfrak{A}$  being defined by

$$(3) \quad cz = (a + vb)(x + vy) = (ax + yobS) + v(aS \cdot y + xb)$$

for  $a, b, x, y$ , in  $\mathbb{C}$ .

In the original form of this paper, the author considered the problem of equivalence in the class of algebras  $\mathfrak{A} = \mathbb{C} + v\mathbb{C}$  with multiplication defined by

$$(4) \quad cz = (a + vb)(x + vy) = (ax + g \cdot ybS) + v(aS \cdot y + xb)$$

for  $a, b, x, y$  in  $\mathbb{C}$  where  $g$  is a fixed element of  $\mathbb{C}$ ,  $g \in \mathfrak{F}$ . The author had shown in [5] that  $\mathfrak{A}$  is a division algebra in case  $g$  is chosen with  $n(g)$  not a square in  $\mathfrak{F}$ ; in particular, such a choice of  $g$  can be made when  $\mathfrak{F}$  is the field  $R$  of rational numbers. R. H. Bruck, the referee of the paper in its original form, suggested a study of the wider class of algebras defined by (3). Theorems 1 and 2 are generalizations of the result in [5] and are due<sup>2</sup> to R. H. Bruck. By their use the class of algebras studied here has been considerably enlarged.<sup>3</sup>

In §2 we shall determine conditions for the equivalence of two alge-

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<sup>1</sup> This modification of the Cayley-Dickson process was originally presented by R. H. Bruck in [2], Theorem 16C, to obtain non-alternative division algebras of orders 4 and 8. Numbers in brackets refer to the references cited at the end of the paper.

<sup>2</sup> Theorem 2 was communicated to the author by Bruck, complete except for the proof of the equivalence of  $\mathbb{C}_*$  and  $\mathbb{C}$ , a fact which Bruck conjectured.

<sup>3</sup> See the comment following the corollary to Theorem 4.

bras  $\mathfrak{A}$ ,  $\mathfrak{A}_*$  of this class. These conditions lead directly to a determination of the automorphisms of  $\mathfrak{A}$ .

**1. A class of division algebras of order 16.** An algebra  $\mathfrak{Q}$  is called a quaternion algebra if  $\mathfrak{Q} = (1, u_2, u_3, u_4)$ ,  $u_4 = u_3u_2$ ,  $u_2^2 = u_2 + \alpha$ ,  $u_3^2 = \beta$ ,  $u_2u_3 = u_3(1 - u_2)$ , where  $\alpha$  and  $\beta \neq 0$  are in  $\mathfrak{F}$ ,  $-4\alpha \neq 1$ . An algebra  $\mathfrak{C}$  is called a Cayley-Dickson algebra if  $\mathfrak{C} = \mathfrak{Q} + u_5\mathfrak{Q}$ , with elements  $x = p + u_5q$ , where  $p, q$  are quaternions and multiplication is defined by  $(p_1 + u_5q_1)(p_2 + u_5q_2) = (p_1p_2 + \gamma q_2 \cdot q_1S) + u_5(p_1S \cdot q_2 + p_2q_1)$  where  $u_5^2 = \gamma \neq 0$  in  $\mathfrak{F}$  and  $S$  is the involution (2) of  $\mathfrak{Q}$ .

Among the well known properties of  $\mathfrak{C}$  which we shall use are that  $\mathfrak{C}$  is an alternative algebra (see [4]), that  $\mathfrak{C}$  is central simple, that (1) holds for  $x$  in  $\mathfrak{C}$ , where

$$(5) \quad x + xS = t(x), \quad x(xS) = (xS)x = n(x)$$

for the involution (2) of  $\mathfrak{C}$ . The norm form  $n(x)$  permits composition—that is,  $n(xy) = n(x)n(y)$  for  $x, y$  in  $\mathfrak{C}$ .

**THEOREM 1.** *Let  $\mathfrak{C}$  be a Cayley-Dickson division algebra over  $\mathfrak{F}$  with involution  $S$ . Let  $\mathfrak{A} = \mathfrak{C} + v\mathfrak{C}$  have multiplication defined by (3), where  $\mathfrak{C}_0$  is chosen so that*

$$(6) \quad n(xoy) = \lambda n(x)n(y), \quad \lambda \neq 0 \text{ in } \mathfrak{F},$$

for all  $x, y$  in  $\mathfrak{C}$ . If  $\lambda$  is not a square in  $\mathfrak{F}$ , then  $\mathfrak{A}$  is a division algebra over  $\mathfrak{F}$ .

A necessary and sufficient condition that  $\mathfrak{A}$  be a division algebra is that

$$(7) \quad ax + yobS = 0, \quad aS \cdot y + xb = 0 \quad (a, b, x, y \text{ in } \mathfrak{C})$$

imply either  $c = 0$  or  $z = 0$ . If we assume on the contrary that (7) hold for nonzero  $c, z$  it follows that  $a, b, x, y$  are all nonzero. Use of (5) then shows (7) essentially equivalent to  $yobS = a[(aS \cdot y)bS]/n(b)$ . Taking norms, we get

$$(8) \quad n(yobS) = [n(a)/n(b)]^2 n(y)n(bS).$$

If  $\lambda$  in (6) is not a square in  $\mathfrak{F}$ , then (8) cannot be satisfied for any  $a, b, y$  in  $\mathfrak{C}$ , and  $\mathfrak{A}$  is a division algebra over  $\mathfrak{F}$ .

In order to complete the proof of Theorem 2 we require two simple lemmas concerning Cayley-Dickson algebras.

**LEMMA 1.** *Let  $\mathfrak{C}$  and  $\mathfrak{C}_*$  be Cayley-Dickson algebras with respective*

unity elements 1 and e and principal equations  $x^2 - t(x)x + n(x)1 = 0$  and

$$(9) \quad x * x - t_*(x)x + n_*(x)e = 0.$$

If  $\mathbb{C}$  and  $\mathbb{C}_*$  are equivalent, the equivalence being given by  $x \leftrightarrow xP$ ,  $x$  in  $\mathbb{C}_*$ ,  $xP$  in  $\mathbb{C}$ , then

$$(10) \quad S_*P = PS$$

where  $S_*$  is the involution

$$(11) \quad S_*: x \leftrightarrow xS_* = t_*(x)e - x$$

of  $\mathbb{C}_*$  and  $S$  is the involution (2) of  $\mathbb{C}$ .

For (9) implies that  $(x * x)P - t_*(x)xP + n_*(x)eP = (xP)^2 - t_*(x)xP + n_*(x)1 = 0$  since  $eP = 1$ . But  $(xP)^2 - t(xP)xP + n(xP)1 = 0$ . Hence

$$(12) \quad t_*(x) = t(xP), \quad n_*(x) = n(xP).$$

But then  $(xS_*)P = [t_*(x)e - x]P = t_*(x)1 - xP = t(xP) - xP = (xP)S$  for all  $x$  in  $\mathbb{C}_*$ , or  $S_*P = PS$ .

LEMMA 2. Let  $\mathbb{C}$  and  $\mathbb{C}_*$  be Cayley-Dickson division algebras over  $\mathfrak{F}$  with norm functions  $n(x)$  and  $n_*(x)$ . If

$$(13) \quad n(x * y) = n(x)n(y), \quad x, y \text{ in } \mathbb{C},$$

then

$$(14) \quad n_*(x) = n(x).$$

We resort to the matrix representation of quadratic forms, regarding  $x$  as a one-rowed matrix, so that the matrix product  $xAx' = n(x)$ , where  $A$  is an  $8 \times 8$  matrix with elements in  $\mathfrak{F}$  (' denoting transpose). Let  $R_y, R_y^*$  be the right multiplications defined by  $xR_y = xy, xR_y^* = x * y$ . Then  $n(x * y) = n(xy)$  or  $xR_y^*AR_y'^*x' = xR_yAR_y'x'$ , from which it follows that  $R_y^*AR_y'^* - R_yAR_y'$  is a skew-symmetric matrix, and  $R_y^*(A + A')R_y'^* = R_y(A + A')R_y'$ . Taking determinants, and noting that  $|A + A'| \neq 0$  since  $\mathbb{C}$  is a division algebra, we obtain  $|R_y^*|^2 = |R_y|^2$  or  $[n_*(y)]^8 = [n(y)]^8$ . Thus  $n_*(y) = \epsilon n(y), \epsilon^8 = 1$ . Take  $y = 1$ . Then  $\epsilon = 1$  and (14) follows.

THEOREM 2. Let  $\mathbb{C}$  be a Cayley-Dickson division algebra over  $\mathfrak{F}$  with unity element 1, and  $\mathbb{C}_o$  be a division algebra with the same elements as  $\mathbb{C}$  and with multiplication  $xoy$ . Then (6) holds if and only if, first,  $\mathbb{C}_o$  is isotopic to  $\mathbb{C}$  (that is, there exist nonsingular linear transformations  $U, V, W$  on  $\mathbb{C}$  such that

$$(15) \quad xoy = (xU \cdot yV)W$$

for  $x, y$  in  $\mathfrak{C}$ ) and, second,<sup>4</sup>

$$(16) \quad n(xT) = n(x)n(1T), \quad T = U, V, W.$$

If  $\mathfrak{A} = \mathfrak{C} + v\mathfrak{C}$ , multiplication defined by (3), (15), (16), and if  $n(1U)n(1V)n(1W)$  is not a square in  $\mathfrak{F}$ , then  $\mathfrak{A}$  is a division algebra over  $\mathfrak{F}$ .

Define two nonsingular linear transformations  $H, J$  on  $\mathfrak{C}$  in the following way: let  $xH = xo1$ ,  $xJ = 1ox$ , for  $x$  in  $\mathfrak{C}$ . Then, if (6) holds,  $n(xH^{-1}o1) = n(x) = \lambda n(xH^{-1})$  and

$$(17) \quad n(xH^{-1}) = n(x)/\lambda.$$

Also  $n(1oxHJ^{-1}) = n(xH) = n(xo1) = \lambda n(x) = \lambda n(xHJ^{-1})$  or

$$(18) \quad n(xHJ^{-1}) = n(x).$$

Let  $\mathfrak{C}_*$  be the isotope of  $\mathfrak{C}_o$  defined by

$$(19) \quad x * y = [xo(yHJ^{-1})]H^{-1}.$$

Then 1 is a unity element for  $\mathfrak{C}_*$ . Moreover,  $n(x * y) = n[xo(yHJ^{-1})]/\lambda = n(x)n(yHJ^{-1}) = n(x)n(y)$ , or (13) holds. That is,  $n(x)$  is a quadratic form permitting composition with respect to the multiplication  $x * y$  of  $\mathfrak{C}_*$ ; hence<sup>5</sup>  $\mathfrak{C}_*$  is a Cayley-Dickson algebra over  $\mathfrak{F}$  and  $n(x)$  is equivalent in  $\mathfrak{F}$  to the norm form  $n_*(x)$  of  $\mathfrak{C}_*$ . (Actually  $n(x)$  is identical with  $n_*(x)$  by (14), but that fact is immaterial at this point of the proof.) Since any two Cayley-Dickson algebras with equivalent norm forms are equivalent,<sup>6</sup> it follows that  $\mathfrak{C}_* \cong \mathfrak{C}$  with

$$(20) \quad x * y = (xU \cdot yU)U^{-1},$$

for  $U$  a nonsingular linear transformation on  $\mathfrak{C}$ . Moreover,  $\mathfrak{C}_o$  is isotopic to  $\mathfrak{C}$ . Let

$$(21) \quad V = JH^{-1}U, \quad W = U^{-1}H.$$

Then (15) follows from (19), (20), (21). Also (16) holds, for  $n(xU) = n_*(x) = n(x)$  by (12) and (14), while  $1U = 1 = n(1U)$ ; also  $n(xV)$

<sup>4</sup> Any nonsingular linear transformation  $T$  satisfying (16) is closely related to a norm-preserving transformation  $U$ . For set  $1T = t$  and  $U = TR_t^{-1}$ . Then  $T = UR_t$  and  $xT = xU \cdot t$ ,  $n(xT) = n(xU)n(t)$ . Hence  $n(xU) = n(x)$ . Conversely if  $T = UR_t$  where  $U$  is norm-preserving, then  $T$  has property (16).

<sup>5</sup> See the principal theorem of [1, p. 161], for the relationship between quadratic forms permitting composition and the norm forms of certain alternative algebras.

<sup>6</sup> [3, p. 777]. The proof in Jacobson's paper is for  $\mathfrak{F}$  of characteristic not two. The author has verified the theorem for arbitrary  $\mathfrak{F}$ , but will not take the space to include a proof here.

$=n(xJH^{-1}U) = n(xJH^{-1}) = n(x)$  by (18) while  $1V = 1 = n(1V)$ , and  $n(xW) = n(xU^{-1}H) = \lambda n(xU^{-1}) = \lambda n(x)$  by (17) while  $n(1W) = \lambda n(1) = \lambda$ .

Conversely, if multiplication in  $\mathbb{C}_o$  is defined by (15) and (16), then (6) holds. For  $n(xoy) = n[(xU \cdot yV)W] = n(xU \cdot yV)n(1W) = n(xU)n(yV)n(1W) = \lambda n(x)n(y)$  where  $\lambda = n(1U)n(1V)n(1W)$ . By Theorem 1, a sufficient condition that  $\mathfrak{A}$  be a division algebra is that  $\lambda$  be not a square in  $\mathfrak{F}$ .

**2. Equivalence in this class of algebras.** The algebras of §1 are quite general, and are at the same time a concrete realization of the hypotheses in the theorems of this section. In order to show that these theorems, which are actually proved for a more general class of division algebras, apply to the algebras described in §1, we note that if  $\mathbb{C}$  is a Cayley-Dickson division algebra over  $\mathfrak{F}$ , and  $\mathfrak{A} = \mathbb{C} + v\mathbb{C}$  has multiplication defined by (3) and (6),  $\lambda$  not a square in  $\mathfrak{F}$ , then, for any nonzero element  $y$  of  $\mathbb{C}$ , the product  $yoyS$  is not in  $\mathfrak{F}$ . For if  $yoyS \in F$ , then  $(yoyS)^2 = n(yoyS) = \lambda n(y)n(yS) = \lambda [n(y)]^2$ , and  $\lambda$  is a square in  $\mathfrak{F}$ , a contradiction.

**THEOREM 3.** *Let  $\mathfrak{A} = \mathbb{C} + v\mathbb{C}$ , with multiplication defined by (3) where  $yoyS$  is not in  $\mathfrak{F}$  for nonzero  $y$  in  $\mathbb{C}$ . Then an element  $z$  of  $\mathfrak{A}$  satisfies a quadratic equation with coefficients in  $\mathfrak{F}$  if and only if  $z$  is in  $\mathbb{C}$ .*

For  $z^2 = (x + vy)^2 = (x^2 + yoyS) + v(xS \cdot y + xy) = t(x)x - n(x) + yoyS + t(x)vy = t(x)z - n(x) + yoyS$ . If  $z^2 - T(z)z + N(z) = 0$  for some  $T(z)$  and  $N(z)$  in  $\mathfrak{F}$ , then  $t(x)z - n(x) + yoyS = T(z)z - N(z)$ , or  $\{t(x) - T(z)\}z = n(x) - yoyS - N(z)$  in  $\mathbb{C}$ . Either  $z$  is in  $\mathbb{C}$ , or  $t(x) - T(z) = 0 = n(x) - yoyS - N(z)$  and  $yoyS = n(x) - N(z)$  in  $\mathfrak{F}$ , contrary to hypothesis. Therefore the only elements of  $\mathfrak{A}$  satisfying quadratic equations are the elements  $z = x$  of  $\mathbb{C}$ .

Let  $\mathfrak{A}_* = \mathbb{C}_* + v' * \mathbb{C}_*$  where  $\mathbb{C}_*$  is a Cayley-Dickson algebra with principal equation (9),  $\mathbb{C}_1$  is a division algebra with the same elements as  $\mathbb{C}_*$  but with multiplication  $[x, y]$ . Let multiplication in  $\mathfrak{A}_*$  be defined by

$$(22) \quad (a + v' * b) * (x + v' * y) = \{a * x + [y, bS_*]\} + v' * (aS_* * y + x * b)$$

for  $a, b, x, y$  in  $\mathbb{C}_*$ , where  $S_*$  is the involution (11).

**THEOREM 4.** *A division algebra  $\mathfrak{A}_* = \mathbb{C}_* + v' * \mathbb{C}_*$  with multiplication defined by (22),  $[y, yS_*]$  not in  $\mathfrak{F}$  for nonzero  $y$  in  $\mathbb{C}_*$ , is equivalent to a division algebra  $\mathfrak{A} = \mathbb{C} + v\mathbb{C}$  with multiplication defined by (3),  $yoyS$  not in  $\mathfrak{F}$  for nonzero  $y$  in  $\mathbb{C}$ , if and only if, first,  $\mathbb{C}_* \cong \mathbb{C}$ , the equivalence being given by  $x \leftrightarrow xP$ ,  $x$  in  $\mathbb{C}_*$ ,  $xP$  in  $\mathbb{C}$ , and, second,  $\mathbb{C}_1 \cong \mathbb{C}_o$  by the*

specific equivalence  $x \leftrightarrow \delta^2 xP$ ,  $x$  in  $\mathfrak{C}_1$ ,  $\delta^2 xP$  in  $\mathfrak{C}_0$ , for some  $\delta \neq 0$  in  $\mathfrak{F}$ . (This second condition may be stated as

$$(23) \quad [x, y]P = \delta^2(xPoyP)$$

for some  $\delta \neq 0$  in  $\mathfrak{F}$ .) The equivalence between  $\mathfrak{A}_*$  and  $\mathfrak{A}$  is  $x + v' * y \leftrightarrow xP + \delta v(yP)$ .

Suppose  $\mathfrak{A}_* \cong \mathfrak{A}$ . Since  $\mathfrak{A}_*$  contains a Cayley-Dickson subalgebra  $\mathfrak{C}_*$  containing all of the elements of  $\mathfrak{A}_*$  which satisfy quadratic equations with coefficients in  $\mathfrak{F}$ , it follows that  $\mathfrak{C}_* \cong \mathfrak{C}$ . Let the equivalence  $H$  between  $\mathfrak{A}_*$  and  $\mathfrak{A}$ ,  $z \leftrightarrow zH$ ,  $z$  in  $\mathfrak{A}_*$ ,  $zH$  in  $\mathfrak{A}$ , have matrix

$$(24) \quad H = \begin{pmatrix} P & Q \\ T & D \end{pmatrix}.$$

Then  $Q=0$ . For if  $a \in \mathfrak{C}_*$ , both  $a$  and  $aH$  satisfy quadratic equations with coefficients in  $\mathfrak{F}$ ,  $aH \in \mathfrak{C}$  and  $Q=0$  in (24). The nonsingularity of  $H$  implies that  $P$  and  $D$  are nonsingular. Moreover, the equivalence  $H$  between  $\mathfrak{A}_*$  and  $\mathfrak{A}$  induces the equivalence  $P$  between  $\mathfrak{C}_*$  and  $\mathfrak{C}$ .

Let  $L_x, L_x^*, R_y, R_y^*$  be the left and right multiplications of  $\mathfrak{C}$  and  $\mathfrak{C}_*$  defined by  $xy = yL_x = xR_y$ ,  $x * y = yL_x^* = xR_y^*$ . Then multiplication is defined in  $\mathfrak{A}$  by  $cz = cR_z$ ,

$$R_z = \begin{pmatrix} R_x & SR_y \\ SL_y^{(o)} & L_x \end{pmatrix}, \quad aL_y^{(o)} = yoa,$$

and in  $\mathfrak{A}_*$  by  $c * z = cR_z^*$ ,

$$R_z^* = \begin{pmatrix} R_x^* & S_*R_y^* \\ S_*L_y^{(1)} & L_x^* \end{pmatrix}, \quad aL_y^{(1)} = [y, a],$$

where  $R_x^* = PR_{xP}P^{-1}$ ,  $L_x^* = PL_{xP}P^{-1}$ , and so on.

The equivalence of  $\mathfrak{A}_*$  and  $\mathfrak{A}$  is given by  $R_z^*H = HR_{zH}$  for all  $z$  in  $\mathfrak{A}_*$ , or

$$(25) \quad \begin{pmatrix} PR_{xP}P^{-1} & S_*PR_yP^{-1} \\ S_*L_y^{(1)} & PL_{xP}P^{-1} \end{pmatrix} \begin{pmatrix} P & 0 \\ T & D \end{pmatrix} = \begin{pmatrix} P & 0 \\ T & D \end{pmatrix} \begin{pmatrix} R_{xP+yT} & SR_yD \\ SL_y^{(o)} & L_{xP+yT} \end{pmatrix}$$

for all  $x, y$  in  $\mathfrak{C}_*$ .

It follows from (25) that  $PR_{xP} + S_*PR_yP^{-1}T = PR_{xP+yT}$  or  $S_*PR_yP^{-1}T = PR_{yT}$ . Then (10) implies that  $PSR_yP^{-1}T = PR_{yT}$  or  $SR_yP^{-1}T = R_{yT}$ . Let  $y = e$  and denote  $eT$  by  $t$ . Since  $eP = 1$ , it follows that  $SP^{-1}T = R_{eT} = R_t$ , and  $T = PSR_t$ . Therefore  $SR_yP^{-1}PSR_t = R_{yPSR_t}$  or  $L_yPSR_t = R_{yPSR_t}$ . By a lemma of Moufang [4, Lemma 1],

$L_{yPS}R_{yPS}R_t = R_{yPS}R_tL_{yPS}$ . Hence  $L_{yPS}R_{yPS}R_t = L_{yPS}R_tL_{yPS}$ , and for all  $y \neq 0$ ,  $R_{yPS}R_t = R_tL_{yPS}$ .

If  $t \neq 0$ , let  $y = tP^{-1}$ . Then  $R_{tS}R_t = R_tL_{tS} = n(t)I = R_tR_{tS}$  and  $L_{tS} = R_{tS}$ ,  $t = \xi$  in  $\mathfrak{F}$ . That is, if  $t \neq 0$ , then  $\xi R_{yPS} = \xi L_{yPS}$ ,  $R_{yPS} = L_{yPS}$ ,  $yPS \in \mathfrak{F}$  for every  $y \neq 0$  in  $\mathbb{C}_*$ , and  $P$  is singular, a contradiction. Hence  $eT = t = 0$  and  $T = PSR_t = 0$  in (24).

With this result and (10), we may write (25) as

$$(26) \quad \begin{pmatrix} PR_xP^{-1} & PSR_{yP}P^{-1} \\ PSP^{-1}L_y^{(1)} & PL_xP^{-1} \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} R_xP & SR_{yD} \\ SL_{yD}^{(o)} & L_xP \end{pmatrix}$$

from which it follows that  $PSR_{yP}P^{-1}D = PSR_{yD}$ , or  $R_{yP}P^{-1}D = R_{yD}$ . Let  $y = e$  and denote  $eD$  by  $d$ . Since  $eP = 1$ , it follows that  $P^{-1}D = R_{eD} = R_d$ , or  $D = PR_d$ . Thus  $R_{yP}P^{-1}PR_d = R_{yP}R_d$  or  $R_{yP}R_d = R_{yPR_d}$ . Let  $y = xP^{-1}$ . Then  $R_xR_d = R_{xd}$  for every  $x$  in  $\mathbb{C}$ . Therefore  $d$  is a scalar  $\delta$  in  $\mathfrak{F}$ , or  $D = \delta P$ ,  $\delta \neq 0$ . Thus

$$H = \begin{pmatrix} P & 0 \\ 0 & \delta P \end{pmatrix}$$

and (26) reduces to three identities and the final relationship  $PSP^{-1}L_y^{(1)}P = \delta^2PSL_{yP}^{(o)}$  or  $L_y^{(1)}P = \delta^2PL_{yP}^{(o)}$  from which (23) follows.

**COROLLARY.** Let  $\mathfrak{A} = \mathbb{C} + v\mathbb{C}$  be a division algebra with multiplication defined by (4),  $g \in \mathfrak{F}$ , and  $\mathfrak{A}_* = \mathbb{C}_* + v' * \mathbb{C}_*$  with multiplication defined by

$$(a + v' * b) * (x + v' * y) = \{ a * x + g_1 * (y * bS_*) \} + v' * (aS_* * y + x * b)$$

for  $a, b, x, y$  in  $\mathbb{C}_*$ , where  $S_*$  is the involution (11) of  $\mathbb{C}_*$  and  $g_1$  is a fixed element of  $\mathbb{C}_*$ ,  $g_1 \in \mathfrak{F}$ . Then  $\mathfrak{A}_* \cong \mathfrak{A}$  if and only if, first,  $\mathbb{C}_*$  is equivalent to  $\mathbb{C}$ , the equivalence being given by  $x \leftrightarrow xP$ ,  $x$  in  $\mathbb{C}_*$ ,  $xP$  in  $\mathbb{C}$ , and, second,  $g_1P = \delta^2g$  for some  $\delta \neq 0$  in  $\mathfrak{F}$ . The equivalence between  $\mathfrak{A}_*$  and  $\mathfrak{A}$  is  $x + v' * y \leftrightarrow xP + \delta v(yP)$ .

For  $[x, y] = g_1 * (x * y)$  and  $xoy = g(xy)$  in Theorem 4. Then (23) becomes  $\{ g_1 * (x * y) \} P = g_1P(xP \cdot yP) = \delta^2g(xP \cdot yP)$ , or simply  $g_1P = \delta^2g$ .

By Theorem 4 we see that the class of algebras described in §1 is actually considerably larger than that defined by (4), which was the class originally studied. For any such algebra is the particular case  $U = I$ ,  $V = I$ ,  $W = L_o$  of (3), (15), (16). Even such a simple variation  $\mathfrak{A}_*$  of  $\mathfrak{A}$  as that in which multiplication is defined by

$$(27) (a + v' * b) * (x + v' * y) = \{ a * x + (y * bS_*) * g_1 \} + v' * (aS_* * y + x * b),$$

$g_1 \in \mathfrak{F}$ , cannot be equivalent to  $\mathfrak{A}$ , since (27) is the case  $U = I$ ,  $V = I$ ,

$W = R_{g_1}^*$  of (3), (15), (16), and  $\mathbb{C}_1$  cannot be equivalent to  $\mathbb{C}_0$  as required by Theorem 4. For  $\mathbb{C}_0$  has a left unity quantity  $g^{-1}$ , while  $\mathbb{C}_1$  has a right unity quantity (the inverse of  $g_1$  with respect to multiplication  $x * y$  in  $\mathbb{C}_*$ ). If  $\mathbb{C}_0 \cong \mathbb{C}_1$ , then  $\mathbb{C}_0$  has a right unity quantity  $h$ , and  $h = g^{-1}$ ,  $\mathbb{C}_0$  has unity quantity  $g^{-1}$ . Then  $x = xog^{-1} = gxg^{-1}$ , or  $xg = gx$  for all  $x$  in  $\mathbb{C}$ ,  $g \in \mathfrak{F}$ , a contradiction. Actually  $\mathbb{C}_1$  is anti-isomorphic to  $\mathbb{C}_0$  in the case  $\mathbb{C}_* \cong \mathbb{C}$  (that is,  $R_y^* = PR_{yP}P^{-1}$ ) and  $g_1P = gS$ , for then  $R_y^{(1)} = R_y^*R_{g_1}^* = PR_{yP}R_{g_1P}P^{-1} = PS^2R_{yP}R_{g_1P}P^{-1} = PSL_{yPS}L_{g_1SP}^{-1} = (PS)L_{yPS}^0(PS)^{-1}$ .

The automorphisms of the division algebras of order 16 over  $\mathfrak{F}$  which are studied in this paper are given directly by the conditions of Theorem 4,  $\mathfrak{A}_* = \mathfrak{A}$ .

**THEOREM 5.** *A nonsingular linear transformation  $H$  on a division algebra  $\mathfrak{A} = \mathbb{C} + v\mathbb{C}$  with multiplication defined by (3),  $yoyS$  not in  $\mathfrak{F}$  for nonzero  $y$  in  $\mathbb{C}$ , is an automorphism of  $\mathfrak{A}$  if and only if  $H$  induces an automorphism  $P$  on  $\mathbb{C}$  and  $\delta^2P$  is an automorphism of  $\mathbb{C}_0$  for some  $\delta \neq 0$  in  $\mathfrak{F}$ . Such an automorphism  $H$  of  $\mathfrak{A}$  has the form  $x + vy \leftrightarrow xP + \delta v(yP)$ .*

**COROLLARY.** *A nonsingular linear transformation  $H$  on a division algebra  $\mathfrak{A} = \mathbb{C} + v\mathbb{C}$  with multiplication defined by (4),  $g \in \mathfrak{F}$ , is an automorphism of  $\mathfrak{A}$  if and only if  $H$  induces an automorphism  $P$  on  $\mathbb{C}$  such that  $gP = \delta^2g$  for some  $\delta \neq 0$  in  $\mathfrak{F}$ . Such an automorphism  $H$  has the form  $x + vy \leftrightarrow xP + \delta v(yP)$ . If  $t(g) \neq 0$ , then  $gP = g$  and either  $\delta = 1$  or  $\delta = -1$ . If  $t(g) = 0$ , then  $\delta^4 = 1$ .*

For  $g_1 = g$  in the corollary to Theorem 4. Moreover,  $t(gP) = t(g)$ ,  $n(gP) = n(g)$ . If  $t(g) \neq 0$ , then  $t(gP) = t(\delta^2g) = \delta^2t(g) = t(g)$  implies that  $\delta^2 = 1$ ,  $\delta = \pm 1$ . If  $t(g) = 0$ , then  $n(gP) = n(\delta^2g) = \delta^4n(g) = n(g)$  implies that  $\delta^4 = 1$ .

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