A NOTE ON LIE GROUPS

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1. **Introduction.** The following theorem, which plays a role in the classification of Lie groups, was first proved by H. Weyl [1, 2]:[1]

THEOREM A. *If G is a real compact connected semi-simple Lie group, then any connected group G' locally isomorphic with G is also compact.*

It is well known and easily seen by considering the simply connected covering group that Theorem A can also be formulated as follows:

THEOREM B. *The fundamental group of a real connected compact semi-simple Lie group is finite.*

In this note we present two proofs of Theorems A and B; one proof uses differential forms, the other, which is somewhat more elementary, is based on differential geometry.[2]

Let then G be a real connected compact Lie group and assume that the fundamental group of G is infinite. We have to prove that G is not semi-simple. We note that for compact groups "semi-simple" means that the center of G is finite [2, p. 282].

2. **Proof by differential forms.** Since for group manifolds the fundamental group and the one-dimensional homology group coincide, our assumption means that the one-dimensional Betti number is not 0. Let Z denote a 1-cycle, which is not homologous to 0 (with rational or real coefficients). By de Rhams theorem there exists an exact differential form ω of degree one such that $\int_Z \omega \neq 0$. It is well known from Cartan's investigations that we can replace ω by a form $\bar{\omega}$ which is invariant under the right and left translations of G. We denote by $a \cdot \theta$ resp. $\theta \cdot b$ the transform of the differential form θ under left resp. right translation so that $a \cdot \theta(x, dx) = \theta(a \cdot x, a \cdot dx)$, where $a \cdot x$ means the group product of the elements a and x of G and $a \cdot dx$ means the image of the vector dx under the left translation by a, and similarly for $\theta \cdot b$. With Haar measure on G we form the expression $\bar{\omega} = \int\int_G a \cdot \omega \cdot b \, da db$; this is an invariant form on G of degree 1. We consider now

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The first proof has also been known to C. Chevalley and G. de Rham for some time, and is given here mainly for completeness' sake.

$$\int_Z \bar{\omega} = \int\int_G \int_Z a\cdot\omega\cdot b \, dadb.$$

By the formula for transformation of integrals we have $\int_Z a\cdot\omega\cdot b = \int_{a\cdot Z}\omega\cdot b$, where $a\cdot Z$ is the image of Z under left translation by a. But $a\cdot Z$ is homotopic to Z since a can be connected with the unit element e by a continuous curve. Therefore $\int_Z a\cdot\omega\cdot b = \int_{a\cdot Z}\omega\cdot b = \int_Z \omega\cdot b$, and by the same reasoning on b we find $\int_Z a\cdot\omega\cdot b = \int_Z\omega$, and therefore finally

$$\int_Z \bar{\omega} = \int\int_G \int_Z \omega \, dadb = \int_Z \omega \cdot \int\int_G dadb = \int_Z \omega \neq 0.$$

The form $\bar{\omega}$ is in particular invariant under the inner automorphisms $a^{-1}\cdot x\cdot a$ of G. Considering $\bar{\omega}$ at the unit element e we have then a nonzero linear function on the tangent space at e which is invariant under the linear transformations of the adjoint group. Since G is compact we can introduce in the tangent space at e an inner product which is invariant under the adjoint group. In a space with an inner product a linear function can be identified with a vector and so $\bar{\omega}$ gives us a vector at e invariant under the adjoint group. (If we write $\bar{\omega}(e, dx) = \sum \alpha_i dx_i$ and assume that the adjoint group is represented by orthogonal matrices, this is simply the vector with components α_i.) But then the one-parameter subgroup in direction of this vector is invariant under the adjoint group also, and lies therefore in the center of G, which shows that G is not semi-simple.

3. **Proof by differential geometry.** The second proof rests on the consideration of geodesics. We assume again that the fundamental group of G is infinite. We introduce in G an invariant differential geometry; this is possible since G is compact; "invariant" means that the right and left translations are isometries. It is well known that the geodesics going through e are the one-parameter subgroups.

Let $\overline{G}$ be the simply connected covering group of G; we introduce the "covering" differential geometry on $\overline{G}$ by requiring that the local isomorphism between G and $\overline{G}$ be an isometry. This differential geometry will also be invariant. Because of the assumption on the fundamental group, $\overline{G}$ is not compact.

As covering space of a compact space, $\overline{G}$ is a "complete" Riemannian space; any two points in it can be connected by a shortest geodesic, that is, by one which realizes the absolute minimum of curve length between the two points (see [**3**, **4**, **5**]). In $\overline{G}$ there exist therefore arbitrarily long geodesic segments which are the shortest connections of their end points. By moving the midpoint of each such segment to $\bar{e}$ (the unit element of $\overline{G}$) by means of a left translation, and by considering the limit of a properly chosen sequence we can find a geodesic $\bar{\gamma}$ through $\bar{e}$, which is a "straight line," that is, which realizes the shortest distance in $\overline{G}$ between any two of its points. We shall prove that $\bar{\gamma}$ belongs to the center of $\overline{G}$.

Consider the image $\gamma = c(\bar{\gamma})$ in G of $\bar{\gamma}$ under the covering mapping $c: \overline{G} \rightarrow G$. The group γ may or may not be closed; the closure of γ is a connected compact Abelian Lie group, therefore a torus group T of a certain dimension. We introduce arclength s $(-\infty < s < +\infty)$ on $\bar{\gamma}$, and write $\bar{\gamma}(s)$ for the point on $\bar{\gamma}$ with parameter value s; we can assume $\bar{\gamma}(0) = \bar{e}$.

We determine now a sequence s_n of values of s, such that $s_n \rightarrow +\infty$, and $c(\bar{\gamma}(s_n)) \rightarrow e$. This is possible since T is compact. From a certain n on we can find points $\bar{e}_n$ in $\overline{G}$ such that (1) $c(\bar{e}_n) = e$ and (2) $d(\bar{\gamma}(s_n), \bar{e}_n) = d(c(\bar{\gamma}(s_n)), e)$ (where we denote by d the distance in G and in $\overline{G}$); this is possible because the covering mapping c is a local isometry. It could happen that $\bar{\gamma}(s_n) = \bar{e}_n$. The points $\bar{e}_n$ are in the center of $\overline{G}$, as (1) shows.

Now let a be any element of $\overline{G}$, and consider the transform $\bar{\delta} = a^{-1} \cdot \bar{\gamma} \cdot a$ of $\bar{\gamma}$; transformation by a being an isometry the parameter s on $\bar{\gamma}$ can also be used as arclength on $\bar{\delta}$. Suppose now that $\bar{\delta}$ is different from $\bar{\gamma}$; then in particular the tangent vectors to $\bar{\gamma}$ and $\bar{\delta}$ at $\bar{e}$ must determine an angle different from zero. Let b^+ denote a point with positive s-value on $\bar{\delta}$, and b^- a point with negative s-value on $\bar{\gamma}$. It is well known that the triangle inequality holds for b^+, b^-, and $\bar{e}$, that is, $d(b^+, b^-) < d(b^+, \bar{e}) + d(b^-, \bar{e})$, provided b^+ and b^- are sufficiently close to $\bar{e}$ (see [6]). We choose b^+ and b^- accordingly; let $d(b^+, \bar{e}) + d(b^-, \bar{e}) - d(b^+, b^-) = \eta$; we have then $\eta > 0$.

We determine n such that $d(\bar{\gamma}(s_n), \bar{e}_n) < \eta/3$; the inequality $d(\bar{\delta}(s_n), \bar{e}_n) < \eta/3$ follows then from the fact that the isometrical transformation by the element a transforms $\bar{\gamma}(s_n)$ into $\bar{\delta}(s_n)$, but has $\bar{e}_n$ as fixed point, since $\bar{e}_n$ belongs to the center of $\overline{G}$. We consider now the following broken path $\bar{\zeta}$: from b^- to b^+ on the shortest geodesic joining those two points, from b^+ to $\bar{\delta}(s_n)$ on $\bar{\delta}$, from $\bar{\delta}(s_n)$ to $\bar{e}_n$ on the shortest geodesic, and from $\bar{e}_n$ to $\bar{\gamma}(s_n)$ on the shortest geodesic. It is clear that the length of $\bar{\zeta}$ is less than the distance between b^- and $\bar{\gamma}(s_n)$ as measured on $\bar{\gamma}$—the difference being at least $\eta/3$. But by construction $\bar{\gamma}$ realizes the shortest distance between any two of its points.

Therefore $\bar{\delta}$ cannot be different from $\bar{\gamma}$. But a being an arbitrary element of $\overline{G}$ this means that $\bar{\gamma}$ is in the center of $\overline{G}$; it follows that the torus T, the closure of γ, is in the center of G, and G is shown not to be semi-simple, which finishes the proof.

References

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