ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

262. P. T. Bateman: *On the representations of a number as the sum of three squares.*

In determining $r_s(n)$, the number of representations of a positive integer n as the sum of s squares, it is known that the singular series $\rho_s(n)$ constructed by Hardy (Trans. Amer. Math. Soc. vol. 21 (1920) pp. 255–284) gives exact results for $3 \leqq s \leqq 8$. For $5 \leqq s \leqq 8$ Hardy proved this by showing that the function $\Psi_s(\tau) = 1+\sum_{n+1}^{\infty}\rho_s(n)e^{\pi i\tau n}$, $\vartheta(\tau) > 0$, has exactly the same behavior under the modular subgroup Γ_3 as the function $\vartheta_3(0|\tau)^s = (\sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2})^s = 1+\sum_{n=1}^{\infty} r_s(n)e^{\pi i\tau n}$. For $s=3$, 4 the double series of partial fractions for $\Psi_s(\tau)$ which Hardy used to establish the modular properties of $\Psi_s(\tau)$ is no longer absolutely convergent, even though the proof is correct formally. For $s=4$ absolute convergence is easily restored by grouping terms, but for $s=3$ this is not possible. In this paper the case $s=3$ is successfully treated by supplementing the Hardy method with a limit process of the kind used by Hecke in defining his generalized Eisenstein series. There are some analytical intricacies in applying the limit process, but no formal difficulties. A particular result is that for n square free, $r_3(n) = C\pi^{-1}n^{1/2}\cdot\sum_{k=1}^{\infty}(-n/k)k^{-1}$, where $C=0$ if $n\equiv 7 \pmod 8$, $C=16$ if $n\equiv 3 \pmod 8$, and $C=24$ if $n \equiv 1, 2, 5, 6 \pmod 8$. (Received July 13, 1946.)

263. Garrett Birkhoff and P. M. Whitman: *Representation theory for certain non-associative algebras.*

It is known that the ways of embedding a Lie algebra L in a linear associative enveloping algebra are all obtained from a "universal" enveloping algebra $A_u(L)$ with infinite basis, by setting a suitable ideal of L equal to zero. It is shown that a corresponding theorem holds for any Jordan algebra J, but that $A_u(J)$ has a finite basis if J does. Particular examples are worked out for Lie and Jordan algebras. If L or J is "solvable," then the finite basis theorem is valid in $A_u(L)$ and $A_u(J)$. (Received July 15, 1946.)

264. A. T. Brauer: *Limits for the characteristic roots of a matrix.*

Let $A=(a_{\kappa\lambda})$ be a matrix with real or complex elements. Set $\sum_{\lambda=1}^{n}|a_{\kappa\lambda}| = R_\kappa$, $\sum_{\kappa=1}^{n}|a_{\kappa\lambda}| = T_\lambda$; $\max_{\kappa=1,2,\dots,n}\{R_\kappa\}=R$, $\max_{\lambda=1,2,\dots,n}\{T_\lambda\}=T$; $R_\kappa - |a_{\kappa\kappa}| = P_\kappa$, $T_\lambda - |a_{\lambda\lambda}| = Q_\lambda$. It is proved that each characteristic root ω_ν of A lies in at least one of the circles $|z-a_{\kappa\kappa}| \leqq P_\kappa$ and in at least one of the circles $|z-a_{\lambda\lambda}| \leqq Q_\lambda$. It follows that $|\omega_\nu| \leqq \min(R, T)$. This generalizes a result of Frobenius for matrices with positive elements and improves results of E. T. Browne, W. V. Parker, and A. B. Farnell. Also $\max|\omega_\nu| = \min(R, T)$ if and only if $R_1 = R_2 = \cdots = R_n$ or $T_1 = T_2 = \cdots = T_n$ and if $\arg(a_{\kappa\lambda}) = \phi + \phi_\kappa - \phi_\lambda$. If these conditions are satisfied, then $\omega = e^\phi \cdot \min(R, T)$ is one of the characteristic roots. Suppose that A has real elements. If for a given m and for each λ one has $|a_{mm} - a_{\lambda\lambda}| > P_m + P_\lambda$, then the circle $|z - a_{mm}| \leqq P_m$ contains one and only one characteristic root; this root is real. If this holds for each m, then all the characteristic roots are real. (Received July 13, 1946.)

265. Ellen F. Buck and R. C. Buck: *A note on finitely-additive measures.*

In the space I of integers, the class $\mathfrak{D}_0$ of sets which are essentially finite unions of arithmetic progressions forms a finitely additive measure class. If S is a countable space and Ω is a countable family of subsets of S which are measurable under a finitely additive measure function, satisfying certain simple restrictions, there exist one-to-one point transformations of S into I which preserve measure in the Carathéodory closure of Ω. This is in a sense a stronger form of a theorem stated by Ulam. The topology in I determined by the class $\mathfrak{D}_0$ is also discussed. (Received July 12, 1946.)

266. R. C. Buck: *The measure-theoretic approach to density.*

Ordinary density, $D(A) = \lim n/a_n$, has some of the properties of a finitely additive measure on the countable space I composed of the positive integers. This paper is devoted to an analysis of a special measure μ obtained by applying the Carathéodory extension to a simple basic measure-density. The measurability and nonmeasurability of certain sets is proved using methods of elementary number theory; in particular, if α is irrational, the set $\{[\alpha n + \beta]\}$ intersects every set of positive measure in an infinite set, and is nonmeasurable. Sequences of sets are used to show that the range of values of μ is the closed unit interval. Generalized limit densities are discussed, using properties of regular summability transforms; it is shown that if A is any set having a positive limit density, determined by a regular transform, and if A does not contain a pair of consecutive integers, then there is a set B, also having a generalized density, such that $A \cup B$ and $A \cap B$ do not. (Received July 12, 1946.)

267. C. J. Everett and H. J. Ryser: *Rational vector spaces.*

Infinite-dimensional rational vectors spaces $V(R)$ are isomorphic if their cardinals are equal. Every cardinal is the cardinal of some $V(R)$. There is no universal $V(R)$. Spaces $V(R)$ with *rational* inner product possess analogues of Schwarz, Bessel, triangle, and Hadamard inequalities, and Schmidt process. Necessary and sufficient conditions that a norm be an inner product norm are given. Inner products in finite-dimensional $V(R)$ are completely determined via the Hasse invariants. Infinite inner product spaces are studied in the countable, and in the separable case. Legendre polynomials arise naturally in the former. Normal algebraic number fields $R(\rho)$ with normal maximal real subfield admit trace $T(\bar{\alpha}\beta)$ as inner product (α, β). A field with basis $\delta_1, \cdots, \delta_n$ so that $T(\bar{\delta}_i\delta_j) = \delta_{ij}$ is called an I-field. The only quadratic I-field is $R(i)$; every normal cubic is; the cyclotomic (p^α) field is if and only if $p = 2$. Dirichlet I-fields are determined. The function $\|\alpha\| = (T(\bar{\alpha}\alpha))^{1/2}$ is a pseudo-valuation of $R(\rho)$, the $\|\ \|$-completion of which is a direct sum of n real or $n/2$ complex fields. (Received June 17, 1946.)

268. J. S. Frame: *On the reduction of the conjugating representation of a finite group.*

Let γ be a column vector whose entries are the g elements γ_i of a finite group G, so arranged that the h_σ elements of a class of conjugate elements C_σ $(\sigma=1, \cdots, r)$ come together. The $g\times g$ permutation matrices $R(\gamma_i)$ and $L(\gamma_i)$ of the right and left regular representations, respectively, are defined by the equations $\gamma\gamma_i=R(\gamma_i)\gamma$, $\gamma_i\gamma=L(\gamma_i)\gamma$. Write $\gamma_i^{-1}\gamma\gamma_i=P(\gamma_i)\gamma$, where $P(\gamma_i)=L(\gamma_i^{-1})R(\gamma_i)$, and call the permutation representation P of degree g, whose matrices are $P(\gamma_i)$, the *conjugating representation* of G. Because of the arrangement of group elements, P is a direct sum of transitive permutation matrix representations P_σ of degrees h_σ, $\sigma=1, \cdots, r$, each of which is the same as a group of permutations on the right cosets of the normalizer N_σ of an element of C_σ. A $g\times g$ "entry" matrix $Z=\|z_{ij}\|$ with unitary orthogonal columns is defined by assigning to the ith row in a specified order the $\sum n_\rho^2=g$ linearly independent entries (or coefficients) for the group element γ_i in a complete set of r nonequivalent irreducible representations Γ_ρ. The principal theorem states that the representation $Z^{-1}PZ$ is the direct sum of the representations $\Gamma_\rho\times\bar{\Gamma}_\rho$. Some illustrations and consequences of this are discussed. (Received July 11, 1946.)

269. S. A. Jennings: *The group ring of a class of infinite nilpotent groups.*

Let G be a nilpotent group with lower central series $G=H_1\supset H_2\supset \cdots \supset H_c \supset H_{c+1}=\{1\}$ such that H_i/H_{i+1}, $i=1, 2, \cdots, c$, is a direct product of a finite number of infinite cyclic groups, and let Γ be the group ring of G over any field of characteristic 0. The structure of Γ is studied, and in particular it is shown that Γ contains a two-sided ideal Δ such that Γ/Δ is isomorphic to the coefficient field and $\bigcap\Delta^k=0$. By taking the powers of the ideal Δ as a system of neighborhoods of 0 a natural topology may be introduced in Γ. If Γ^*, Δ^* are the completions of Γ, Δ in this topology then Δ^* is the Jacobson radical of Γ^*. The methods of this paper are similar to those of an earlier investigation of the group ring of a p-group over a field of characteristic p (Trans. Amer. Math. Soc. vol. 50 (1941) pp. 175–185). (Received July 15, 1946.)

270. B. E. Meserve: *Inequalities of higher degree.*

A system composed of a finite number of inequalities of the form $0<f$ where f is a real-valued function of real variables $x_1, x_2, \cdots, x_p$ has as *content* all sets of x's (or points) for which $0<f$ is satisfied. Bounds are determined for the number k of segments, $\alpha<x<\beta$, in the content of a single inequality $0<f$ where f is a polynomial and for the number K of segments in the content of a finite system $0<g_i(x)$. The content of a single polynomial inequality in two indeterminates is discussed. A system $0<b_iE_i$ $(i=1, 2, \cdots, m)$ where E_i is the elementary symmetric polynomial of degree i in $x_1, x_2, \cdots, x_m$ has a solution if and only if $0\neq b_1b_2 \cdots b_m$. General solutions are given for several particular systems $0<b_iE_i$. (Received July 3, 1946.)

271. C. N. Moore: *On the infinitude of prime pairs.*

In his address entitled *Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zetafunktion*, delivered at the Fifth International Congress of Mathematicians (Cambridge, 1912), Edmund Landau mentioned four problems which he regarded as "unangreifbar beim gegenwartigen Stände der Wissenschaften." One of these problems was the question of the infinitude of prime pairs.

It therefore seems desirable to place on record any attack on this problem that gives promise of leading to a solution. In the present communication the problem is solved under the assumption of an unproved property of the functions corresponding to certain types of Dirichlet's series. The Dirichlet's series in question are similar to those that correspond to the functions $(L'/L)(s, \chi)$, arising in the discussion of the number of primes in an arithmetic progression, and the property assumed for the related functions is one possessed by the functions $(L'/L)(s, \chi)$. Accordingly the assumption, which concerns the asymptotic behavior of the related functions on certain curves asymptotic to the axis of imaginaries, appears at least equally reasonable with the classical Riemann hypothesis regarding the roots of the zeta-function. (Received July 2, 1946.)

272. Irma R. Moses: *On the representation, in the ring of p-adic integers, of a quadratic form in n variables by one in m variables.*

The following theorem is proved: Let S and T be symmetric, nonsingular, rationally integral matrices, of dimensions m by m and n by n respectively. If there exist a real transformation and transformations in all p-adic rings, each taking S into T, then there exists a rational transformation without essential denominator, taking S into T. This theorem was proved by C. L. Siegel (Amer. J. Math. vol. 63 (1941) pp. 678–680) in case $m=n$. A considerable portion of his proof must be modified in case $m\neq n$. The modification consists principally in establishing the hypothesis of another theorem of Siegel (Ann. of Math. vol. 36 (1935) p. 536), which gives a relation between p-adic transformations taking S into T; or, more specifically, it consists in showing that a certain determinant is nonvanishing. In doing this, the author uses canonical forms for T in the p-adic rings. In the case of the 2-adic ring, canonical forms were obtained from some results of B. W. Jones (Duke Math. J. vol. 9 (1942) pp. 726–727) and Gordon Pall (Quart. J. Math. Oxford Ser. vol. 6 (1935) pp. 35–38). (Received July 16, 1946.)

273. C. E. Rickert: *The singular elements of a Banach algebra.*

An element of a B-algebra $\mathcal{A}$ is said to be *regular* provided it possesses a two-sided inverse, otherwise it is said to be *singular*. If there exists $\{z_n\} \subset \mathcal{A}$ such that $\|z_n\|=1$ and either $xz_n \to 0$ or $z_n x \to 0$, then x is called a *generalized null-divisor* (g.n.d.) (Silov, C.R. (Doklady) Acad. Sci. URSS. vol. 22 (1939) pp. 7–10). If x is a g.n.d., then it is necessarily singular in every B-algebra which contains $\mathcal{A}$. If every singular element of $\mathcal{A}$ is a g.n.d., then $\mathcal{A}$ is said to be *proper*. The class P of all elements of $\mathcal{A}$, not g.n.d., is an open set and the component of P which contains the identity element is a group under multiplication. This generalizes a result of M. Nagumo (Jap. J. Math. vol. 13 (1936) pp. 61–80). It follows that, if P is connected, $\mathcal{A}$ is proper. $\mathcal{A}$ will also be proper if it possesses a certain general adjoint operation. If $\mathcal{A}$ is commutative and its norm satisfies the condition $\|x^2\|=\|x\|^2$, then it can be embedded in a B-algebra $\mathcal{A}'$ which is proper and in which every element of $\mathcal{A}$, not a g.n.d., is regular. (Received July 13, 1946.)

274. R. M. Robinson: *Primitive recursive functions.*

Recursion with one parameter has the form $F(u, 0)=A(u)$, $F(u, x+1)=B(u, x, F(u, x))$. If $B(u, x, y)$ does not depend on the parameter u, the scheme is called iteration; if it does not depend on x, pure recursion; and if it depends only on y, pure iteration. It is shown that all primitive recursive functions, that is, all arithmetical functions which can be obtained from the identity, zero, and successor functions, by repeated substitutions and recursions with any number of parameters, can be defined using recursion only in the special form of iteration with one parameter. It is shown further that still more special schemes are sufficient if certain adjunctions are made to the initial functions: pure recursion with one parameter, if the predecessor function is adjoined; pure iteration with one parameter, if the characteristic function of squares is adjoined; recursion with no parameter, by adjoining sum and the characteristic function of squares, or absolute difference; pure recursion with no parameter, by adjoining sum and excess over the preceding square, or absolute difference and the characteristic function of squares. In the last case, sum and the characteristic function of squares would not be sufficient to adjoin. (Received July 11, 1946.)

275. R. D. Schafer: *Concerning automorphisms of non-associative algebras.*

N. Jacobson pointed out (*A note on non-associative algebras*, Duke Math. J. vol. 3 (1937) pp. 544–548) that an automorphism S of a non-associative algebra $\mathfrak{A}$ determines in a natural way an automorphism Σ of the (associative) transformation algebra $T(\mathfrak{A})$. Let $\mathfrak{A}$ have unity element 1. With a suitable multiplication defined, the set of residue classes of $T(\mathfrak{A})$ modulo $\mathfrak{N}$, the right ideal of transformations $N \in T(\mathfrak{A})$ annihilating 1, forms an algebra equivalent to $\mathfrak{A}$. Then $S \rightarrow \Sigma$ defines a natural isomorphism between the automorphism group $\mathfrak{G}$ of $\mathfrak{A}$ and the group $\mathfrak{H}$ of all automorphisms Σ of $T(\mathfrak{A})$ under which $\mathfrak{N}$ and the right multiplication space $R(\mathfrak{A})$ are their own images. The study of $\mathfrak{G}$ reduces to that of a well-defined subgroup $\mathfrak{H}$ of the automorphism group of the associative algebra $T(\mathfrak{A})$. (Received July 15, 1946.)

276. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* III.

The quasi-determinant $\Delta(\lambda; \alpha) = |\lambda_{ij} \cdot \alpha_i\alpha_j|$ $(i, j = 1, 2, \cdots, n+1; n = 1, 2, \cdots)$ is "modular" if the configurational sets of dyads A_{ij} in its expansion are subject to equivalence relations; $\Delta(\lambda; \alpha)$ is then reduced relatively to the latter. This reduction may be effected (1) corresponding to the reduction (by equivalence relations) of the symmetric group G of degree $n+1$ to any subgroup H of G (2) by assuming that two A_{ij} are equivalent if they occur in a subgroup H of G or in the same co-set of G relatively to H. Then if $G = H + \sum H \cdot s_i = H + \sum s_i' \cdot H$ and if I, $s_i(I, s_i')$ are elements of the cyclic group of order $m+1$, numerical expressions result from replacing configurational sets corresponding to I, $s_i(I, s_i')$ by roots of the equation $s^{m+1} = 1$ with $\alpha_1\alpha_1 \cdot \alpha_2\alpha_2 \cdots \alpha_{n+1}\alpha_{n+1} = 1$. For $m = 1$, $\Delta(\lambda; \alpha)$ then reduces to a determinant. Another type of modular quasi-determinants is obtained by subjecting the dyads $\alpha_i\alpha_j$ in $\Delta(\lambda; \alpha)$ to equivalence relations. For example, by assuming (1) $\alpha_i\alpha_j = \alpha_k\alpha_l$ if and only if $\alpha_i\alpha_j$ and $\alpha_k\alpha_l$ occur in the same A_{ij} of a regular group, and (2) $\lambda_{ij} = \lambda_{kl}$ if and only if $\alpha_i\alpha_j = \alpha_k\alpha_l$, a modular "group quasi-determinant" is obtained analogous to the group determinant discussed by Frobenius and others. (Received July 11, 1946.)

277. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* IV.

The quasi-determinant $\Delta(\lambda; \alpha)$ in expanded form is interpreted as a special case of a hypercomplex number with group elements as units in a group algebra. More generally, a representation of a group algebra is based on the addition and multiplication of linear functions of ordered dyads $X_{m+1}(\alpha)=\sum x_j\alpha_1\alpha_j$ $(j=1, 2, \cdots, m+1)$ with coefficients x_j in a suitable domain D (field F) relatively to any given abstract group G of order $m+1$ $(m=1, 2, \cdots)$ represented as a regular group of configurational sets of dyads on $m+1$ elements. Two dyads $\alpha_i\alpha_j$ and $\alpha_k\alpha_l$ are then equivalent if and only if they occur in the same configurational set of dyads. Multiplication is determined by $\alpha_i\alpha_j\times\alpha_j\alpha_k=\alpha_i\alpha_k$ and by the preceding equivalences. Other instances of dyadic representation of linear algebras are given by two examples: 1. $X_2(\alpha)=x_1\alpha_1\alpha_1+x_2\alpha_1\alpha_2$ with equivalences $\alpha_1\alpha_1=\alpha_2\alpha_2$, $\alpha_1\alpha_2=-\alpha_2\alpha_1$. 2. $X_4(\alpha)=\sum x_j\alpha_1\alpha_j$ $(j=1, 2, 3, 4)$ with equivalences corresponding to those given by the author (Math. Ann. vol. 69 p. 584). In both examples multiplication is determined by $\alpha_i\alpha_j\times\alpha_j\alpha_k=\alpha_i\alpha_k$ and by the associated equivalences. (Received July 11, 1946.)

278. M. C. Sholander: *On the existence of the inverse operation in certain spaces.*

In a set S of elements $x, y, \cdots$ which admits a binary operation—here denoted by multiplication—an element a will be called regular if both (i) $ax=ay$ implies $x=y$ and (ii) $xa=ya$ implies $x=y$. An element a will be called proper if for each element b in S there exist unique solutions x and y in S for the equations $ax=b$ and $ya=b$. It is well known that if the multiplication is commutative and associative S can be imbedded in a space S' of the same type in such a way that all elements regular in S are proper in S'. In this paper it is shown the imbedding process can also be carried out in case the multiplication is one satisfying the alternation law $(ab)(cd)=(ac)(bd)$ and in case the regular elements of S are closed under multiplication. Thus if all elements of S are regular, S' is a quasi-group of a type studied, for example, by D. C. Murdoch (Trans. Amer. Math. Soc. vol. 49 (1941) pp. 392–409) and R. H. Bruck (Trans. Amer. Math. Soc. vol. 55 (1944) pp. 19–52). Various conditions which insure the necessary closure property in S are given in the paper. (Received July 26, 1946.)

279. J. M. Thomas: *Eliminants.*

If $R(a, b)$ denotes the resultant to two polynomials $x(t)$, $y(t)$ whose constant terms are a, b, the polynomial $R(a-x, b-y)$ in the two indeterminates x, y is the *eliminant* $E(x, y)$ of $x(t)$, $y(t)$. This paper (i) proves $E(x, y)=f^k$, where f is an irreducible polynomial and k is a positive integer; (ii) proves $E(x, y)$ is reducible $(1<k)$ if and only if $x(t)$, $y(t)$ are also polynomials in a second parameter which is itself a polynomial of degree at least two in t; (iii) expresses in terms of $E(x, y)$ algebraic conditions that a single polynomial $y(t)$ be a polynomial in $x(t)$ of degree k, where $1<k<\deg y$ (these last polynomials have been called by Ritt composite polynomials, Trans. Amer. Math. Soc. vol. 23 (1922) pp. 51–66). (Received July 27, 1946.)

280. J. H. M. Wedderburn: *Note on Goldbach's theorem.*

It is shown in this note that, if p and q are primes and $r=(p+q)/2$ is a prime, then $p-q$ is a multiple of 12 unless r has the form $2p-3$ or, when unity is reckoned as a prime, also $r=2p-1$. The proof is elementary and depends on reducing modulo 12. Similar statements apply if q is replaced by $-q$. (Received July 30, 1946.)

Analysis

281. R. P. Agnew: *Methods of summability which evaluate sequences of zeros and ones summable C_1.*