ON ANALYTIC FUNCTIONS WITH BOUNDED CHARACTERISTIC

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A function $f(re^{i\phi})$, regular within the unit circle, is called a function with bounded characteristic if

$$\lim_{r\to 1}\int_0^{2\pi}\log^+\left|f(re^{i\phi})\right|d\phi$$

is bounded, where $\log^+\left|f(re^{i\phi})\right|=\max\left(\log\left|f(re^{i\phi})\right|,\ 0\right)$. If $f(z)$ is a function with bounded characteristic, then

$$\lim_{r\to 1} f(re^{i\phi}) = f(e^{i\phi})$$

exists almost everywhere [1].[2]

In the first part of this paper we prove the following:

THEOREM I. *Let* $\{f_n(z)\}$ $(n=1, 2, 3, \cdots)$ *and* $f(z)$ *be functions with bounded characteristics, let*

$$(1)\qquad \begin{aligned} \log A_n &= \lim_{r\to 1}\int_0^{2\pi}\log^+\left|f_n(re^{i\phi})\right|d\phi,\\ \log A &= \lim_{r\to 1}\int_0^{2\pi}\log^+\left|f(re^{i\phi})\right|d\phi, \end{aligned}$$

and

(2) $\left|f(e^{i\phi})-f_n(e^{i\phi})\right|<m_n,$ *for* $\phi\in E_n$, *and let* μ_n *be the measure of* E_n.

If

$$(3)\qquad \lim_{n\to\infty} m_n^{\mu_n} = 0,$$

and for every positive σ *there exists a positive integer* n_σ *such that*

$$(4)\qquad A_n < m_n^{-\sigma\mu_n}\quad \text{for}\quad n > n_\sigma,$$

then the sequence $\{f_n(z)\}$ *tends uniformly to* $f(z)$ *in any closed domain interior to the unit circle.*

Received by the editors January 8, 1946.

[1] Deceased December 23, 1945.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

Special cases of this theorem are the following:

A. If the numbers A_n are bounded, we need only the condition (3), as (4) follows from (3) in this case.

B. If the numbers A_n are bounded, $E_1 = E_2 = \cdots = E_n = \cdots$ and $\mu_1 > 0$, we need only the condition $\lim_{n\to\infty} m_n = 0$ instead of (3) and (4). This case was proved by Ostrowski [1].

C. If $E_1 = E_2 = \cdots = E_n = \cdots$ and $\mu_1 > 0$, we can replace (3) and (4) by the following two conditions: $\lim_{n\to\infty} m_n = 0$, and for every positive σ there exists a positive integer n_σ such that $A_n < m_n^{-\sigma}$ for $n > n_\sigma$. This case was proved by Milloux [2] under the less general assumptions that the functions $f_n(z)$ and $f(z)$ are bounded, and that $E_1 = E_2 = \cdots = E_n = \cdots$ is an arc (α, β) of the unit circle on which the functions $f_n(z)$ and $f(z)$ are continuous.

In the second part of this paper we prove the following theorem.

THEOREM II. *Let*

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n_k} \tag{5}$$

be a function with bounded characteristic and let

$$\frac{n_{k+1}}{n_k} \geqq q > 1 \qquad (k = 1, 2, \cdots); \tag{6}$$

then

$$\sum_{k=1}^{\infty} |c_k|^2 \tag{7}$$

converges.

This theorem generalizes a theorem of R. Paley [3], which proved that if $f(z)$ has an expansion (5) where the n_k satisfy (6) and

$$\lim_{r\to 1} \int_0^{2\pi} |f(re^{i\phi})|\, d\phi \tag{8}$$

is bounded, then the series (7) converges. From $\log^+|f(z)| < |f(z)|$ it follows that each function which satisfies (8) is of bounded characteristic. Therefore, Paley's theorem is a consequence of Theorem II. We shall show that Theorem II gives a negative answer to the following question asked by Bloch [4]: If $f(z)$ is a function with bounded characteristic, must the derivative of $f(z)$ have the same property?

1. **Proof of Theorem I.** We put $\phi_n(z) = f(z) - f_n(z)$ and $\phi_n(e^{i\phi}) = f(e^{i\phi}) - f_n(e^{i\phi})$. We have

$$\begin{aligned}\log^+ |\phi_n(z)| &= \log^+ |f(z) - f_n(z)| \leqq \log^+ (|f(z)| + |f_n(z)|) \\ &\leqq \log^+ |f(z)| + \log^+ |f_n(z)| + \log 2.\end{aligned} \tag{9}$$

Let $\{r_k\}$ be an increasing sequence of positive numbers with

$$\lim_{k\to\infty} r_k = 1.$$

By Jensen's formula we have, for $r<r_k$,

$$\begin{aligned}\log |\phi_n(re^{i\phi})| + \sum_{\mu=1}^{m} \log \left| \frac{r_k^2 - \bar{a}_\mu r e^{i\phi}}{r_k(re^{i\phi} - a_\mu)} \right| \\ = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r_k^2 - r^2) \log |\phi_n(r_k e^{i\theta})|}{r_k^2 - 2r_k r \cos(\phi - \theta) + r^2} d\theta\end{aligned} \tag{10}$$

where $a_1, a_2, \cdots, a_m$ are the zeros of $\phi_n(z)$ within the circle of radius r_k. As the second term on the left side of (10) is positive we have

$$2\pi \log |\phi_n(re^{i\phi})| \leqq \int_0^{2\pi} \frac{(r_k^2 - r^2) \log |\phi_n(r_k e^{i\theta})| \, d\theta}{r_k^2 - 2r_k r \cos(\phi - \theta) + r^2}. \tag{11}$$

As $\lim_{k\to\infty}\phi_n(r_k e^{i\phi}) = \phi_n(e^{i\phi})$ for $\phi \in E_n$, there exists by Egoroff's theorem, for every positive δ_1, a set $E_n' < E_n$ $(n=1, 2, \cdots)$ such that

$$\mu(E_n') > \mu_n - \delta_1, \tag{12}$$

and

$$\lim_{k\to\infty} \phi_n(r_k e^{i\phi}) = \phi_n(e^{i\phi}) \tag{13}$$

uniformly in E_n'.

Because of (3) we can assume $m_n < 1$ $(n=1, 2, \cdots)$; hence it follows from (2) and (13), that

$$|\phi_n(r_k e^{i\phi})| < m_n + \delta_2 < 1, \tag{14}$$

for $\phi \in E_n'$ and for sufficiently large k, where δ_2 is arbitrarily small. We denote by E_n'' the complement of E_n' with respect to $(0, 2\pi)$. Then (11) takes the form

$$\begin{aligned}2\pi \log |\phi_n(re^{i\phi})| \leqq \int_{E_n''} \frac{(r_k^2 - r^2) \log^+ |\phi_n(r_k e^{i\theta})| \, d\theta}{r_k^2 - 2r_k r \cos(\phi - \theta) + r^2} \\ + \int_{E_n'} \frac{(r_k^2 - r^2) \log |\phi_n(r_k e^{i\theta})| \, d\theta}{r_k^2 - 2r_k r \cos(\phi - \theta) + r^2}.\end{aligned}$$

As the first integral on the right side is positive and the second negative, we have, because of (1), (9) and (14),

$$2\pi \log |\phi_n(re^{i\phi})| \leqq \frac{r_k+r}{r_k-r}\int_{E_n''} \log^+ |\phi_n(re^{i\theta})|\, d\theta + \frac{r_k-r}{r_k+r}\int_{E_n'} \log |\phi_n(re^{i\theta})|\, d\theta$$

$$\leqq \frac{r_k+r}{r_k-r} \log 2AA_n + \frac{r_k-r}{r_k+r}(\mu_n-\delta_1)\log(m_n+\delta_2),$$

for sufficiently large k. We get, therefore, for $\lim_{k\to\infty} r_k = 1$, as δ_1 and δ_2 are arbitrarily small,

$$2\pi \log |\phi_n(re^{i\phi})| \leqq \frac{1+r}{1-r}\log 2AA_n + \frac{1-r}{1+r}\mu_n \log m_n,$$

or

$$(15) \qquad |\phi_n(re^{i\phi})| = (2AA_n)^{(1+r)/2\pi(1-r)} m_n^{\mu_n(1-r)/2\pi(1+r)}.$$

Let B be any closed domain within the unit circle. There exists an $r'<1$, such that the interior of the circle with radius r' contains B. Now we choose

$$(16) \qquad \sigma < \left(\frac{1-r'}{1+r'}\right)^2.$$

If we put $A'=\max(A, 1)$ and $A_n'=\max(A_n, 1)$ we get, for $re^{i\phi}\in B$, by use of (4),

$$|\phi_n(re^{i\phi})| \leqq (2A')^{(1+r')/2\pi(1-r')} m_n^{(\mu_n/2\pi)((1-r')/(1+r')-\sigma(1+r')/(1-r'))}$$

for sufficiently large n.

As, because of (16), $(1-r')/(1+r')-\sigma(1+r')/(1-r')>0$ and since, by assumption, $\lim_{n\to\infty} m_n^{\mu_n}=0$, we have proved the theorem.

2. **Proof of Theorem II.** For the proof of Theorem II we need the following two theorems, the first due to Hardy and Littlewood [5], the second to Zygmund [6].

THEOREM A. *If* $f(z)=\sum_{k=1}^{\infty} c_k z^{n_k}$ and $n_{k+1}/n_k \geqq q > 1$ $(k=1, 2, \cdots)$, *then the existence of* $\lim_{r\to 1} f(re^{i\phi_0}) = f(e^{i\phi_0})$ *implies the convergence of* $\sum_{k=1}^{\infty} c_k e^{in_k\phi_0}$.

THEOREM B. *If the trigonometric series*

$$\sum_{k=1}^{\infty} (a_k \cos n_k\theta + b_k \sin n_k\theta) \qquad (n_k + 1/n_k \geqq q > 1)$$

converges in a set of positive measure, then the series

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

converges.

As $f(z)$ is a function with bounded characteristic

$$\lim_{r \to 1} f(re^{i\phi}) = f(e^{i\phi})$$

exists almost everywhere. If we put $c_k = a_k - ib_k$ then by Theorem A the series

$$\sum_{k=1}^{\infty} (a_k - ib_k)(\cos n_k\phi + i \sin n_k\phi) = \sum_{k=1}^{\infty} (a_k \cos n_k\phi + b_k \sin n_k\phi) + i \sum_{k=1}^{\infty} (a_k \sin n_k\phi - b_k \cos n_k\phi)$$

converges almost everywhere. Therefore, by Theorem B, the series

$$\sum_{k=1}^{\infty} a_k^2 + b_k^2 = \sum_{k=1}^{\infty} | c_k |^2$$

converges.

Now we shall show that there exist bounded functions, whose derivatives are functions whose characteristics are not bounded.

The radius of convergence of the power series

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{n_k}}{n_k} \qquad \left(\frac{n_{k+1}}{n_k} \geqq q > 1\right)$$

is equal to 1 and the function $f(z)$ is bounded within the unit circle. The derivative

$$f'(z) = \sum_{k=1}^{\infty} z^{n_k - 1}$$

is, by Theorem II, a function whose characteristic is not bounded.

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ON THE $(C, 1)$ SUMMABILITY OF CERTAIN RANDOM SEQUENCES

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It is known [1][1] that if a sequence $\{a_n\}$ $(n=1, 2, \cdots)$ of real numbers is summable $(C, 1)$ to a value α, and if $\sum a_n^2/n^2 < \infty$, then almost all the subsequences of $\{a_n\}$ are summable $(C, 1)$ to α. It will be shown that this statement continues to hold if "almost all" is replaced by "with probability 1" and "subsequences" by the more general term "product sequences," the meaning of which will be defined in the next paragraph. The only analytic tool used is the strong law of large numbers [2]: if $\{y_n\}$ is a sequence of independent random variables with expected values $E(y_n)=0$ and $E(y_n^2)=b_n^2$, for which $\sum b_n^2/n^2 < \infty$, then with probability 1 the sequence $\{y_n\}$ is summable $(C, 1)$ to the value 0.

DEFINITION. Let $\{a_n\}$ be a sequence of constants and let $\{x_n\}$ be a sequence of random variables such that the values of each x_n are non-negative integers. For every n let $k(n)$ be the least positive integer m such that

$$(1) \qquad \sum_{1}^{m} x_i \geqq n,$$

Presented to the Society, September 17, 1945; received by the editors January 23, 1946.

[1] Numbers in brackets refer to references listed at end of paper.