DERIVATIVES OF COMPOSITE FUNCTIONS

JOHN RIORDAN

1. Introduction. The object of this note is to show the relation of the Y polynomials of E. T. Bell [1],[1] first to the formula of diBruno for the nth derivative of a function of a function, then to the more general case of a function of many functions. The subject belongs to the algebra of analysis in the sense of Menger [4]; all that is asked is the relation of the derivative of the composite function to the derivatives of its component functions when they exist and no questions of analysis are examined.

2. Function of a single function. Following Dresden [3], take the composite function in the form:

$$F(x) = f[g(x)]; \tag{1}$$

and for convenience write:

$$D_x^s F(x) \equiv F_s, \qquad [D_u^s f(u)]_{u=g(x)} \equiv f_s, \qquad D_x^s g(x) \equiv g_s,$$

with $D_x = d/dx$.

Then, the first few derivatives of $F(x)$ are as follows:

$$F_1 = f_1 g_1, \qquad F_2 = f_1 g_2 + f_2 g_1^2, \qquad F_3 = f_1 g_3 + 3 f_2 g_2 g_1 + f_3 g_1^3.$$

If these are generalized to

$$F_n = \sum_{i=1}^{n} F_{n,i} f_i \tag{2}$$

the coefficients $F_{n,i}$ are dependent only on the derivatives g_1 to g_i, and hence may be determined by specialization of f. A convenient choice used by Schlömilch [6] is $f(g) = \exp(ag)$, so that

$$f_i = a^i \exp ag$$

and

$$e^{-ag} F_n = e^{-ag} D_x^n e^{ag} = \sum_{i=1}^{n} F_{n,i}(g_1, \cdots, g_i) a^i, \tag{3}$$

a generating identity for the $F_{n,i}$, closely related to the definition equation of the Y polynomials (Bell, loc. cit. p. 269), namely

Received by the editors February 15, 1946.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$$e^{-y}D_x^n e^y = Y_n(y_1, \cdots, y_n), \qquad y_s = D_x^s y.$$

Indeed

$$Y_n(ag_1, \cdots, ag_n) = \sum_{i=1}^{n} F_{n,i}(g_1, \cdots, g_i)a^i \tag{4}$$

and, by combination of this with (2),

$$F_n = Y_n(ag_1, \cdots, ag_n), \qquad (a^i \equiv f_i). \tag{5}$$

For concreteness, the first few Y's are listed as follows:

$$Y_0 = 1, \qquad Y_1 = y_1 = ag_1, \qquad Y_2 = y_2 + y_1^2 = ag_2 + a^2 g_1^2,$$
$$Y_3 = y_3 + 3y_2y_1 + y_1^3 = ag_3 + 3a^2 g_2 g_1 + a^3 g_1^3.$$

Recurrence and other relations for the F_n—before the identification $a^i = f_i$—may then be taken from Bell with slight changes for the change in arguments. The more important ones are as follows:

$$\begin{aligned} F_{n+1} &= (ag_1 + D_x)F_n \\ &= ag(F + g)^n = \sum_{i=0}^{n} \binom{n}{i} ag_{i+1}F_{n-i} \\ &= \left(ag_1 + \sum_{i=1}^{n} g_{i+1}\frac{\partial}{\partial g_i}\right)F_n, \end{aligned} \tag{6}$$

$$\exp tF = \exp a[\exp tg - g_0], \tag{7}$$

where the last is symbolic and equivalent to

$$F_0 + tF_1 + t^2F_2/2! + \cdots = \exp a[tg_1 + t^2g_2/2! + \cdots],$$

and to diBruno's formula

$$F_n = \sum \frac{n!a^\sigma}{s_1! \cdots s_n!}\left(\frac{g_1}{1!}\right)^{s_1} \cdots \left(\frac{g_n}{n!}\right)^{s_n} \tag{8}$$

with summation over all non-negative integral solutions of $n = \sum is_i$ and $\sigma = \sum s_i$.

Instances of (7) with special choices of g may be used for verification of the numerical coefficients of F_n. Thus, with $g_i = 1$, all i,

$$\exp tF = \exp a(e^t - 1),$$

and

$$F_n = \sum S_{i,n}a^i, \qquad g_i = 1, \tag{9}$$

with $S_{i,n}=\Delta^i 0^n/i!$ the Stirling number of the second kind, showing that the sum of numerical coefficients of $F_{n,i}$ is $S_{i,n}$. Again with $g_1=0$, $g_i=1$, $i>1$,

$$\exp tF = \exp a(e^t - 1 - t),$$

and

$$F_n = \sum C_{i,n} a^i, \qquad g_1 = 0,\ g_i = 1,\ i > 1, \tag{10}$$

with $C_{i,n}$ an associated Stirling number with recurrence

$$C_{i,n+1} = iC_{i,n} + nC_{i-1,n-1}.$$

These numbers have a combinatorial interpretation similar to that for $S_{i,n}$, namely, $C_{i,n}$ is the number of rhyme schemes of n verses with i rhymes, such that each rhyme appears at least twice. This kind of verification of course may be extended at will. These results are in agreement with results of Wall [8] and Opatowski [5].

3. **Function of many functions.** The essentials of the general case are in the two function case

$$F(x) = f[g(x), h(x)]. \tag{11}$$

If we write

$$f_{ij} = \left[\frac{\partial^i}{\partial u^i} \frac{\partial^j}{\partial v^j} f(u, v) \right]_{u=g(x),\, v=h(x)}$$

the first two derivatives are

$$F_1 = f_{01}g_1 + f_{10}h_1,$$
$$F_2 = f_{20}g_1^2 + 2f_{11}g_1h_1 + f_{02}h_1^2 + f_{10}g_2 + f_{01}h_2,$$

and in general

$$F_n = \sum_i \sum_j f_{ij} F_{n,ij}(g_1 \cdots g_i;\ h_1 \cdots h_j).$$

Then as before the generating function for coefficients $F_{n,ij}$ is $\exp(-ag+bh)D_x^n \exp(ag+bh)$ and

$$F_n = Y_n(ag_1 + bh_1, \cdots, ag_n + bh_n), \qquad a^ib^j = f_{ij}, \tag{12}$$

$$\exp tF = \exp\left[a(\exp tg - g_0) + b(\exp th - h_0)\right]. \tag{13}$$

The last shows that (cf. Bell [1, equations 4.9 and 7.5])

$$F_n = (G + H)^n = \sum \binom{n}{i} G_{n-i} H_i \tag{14}$$

with

$$G_n = Y_n(ag_1, \cdots, ag_n), \qquad H_n = Y_n(bh_1, \cdots, bh_n).$$

The numerical coefficients associated with Y_n then reappear here and in the general case, as noted by Opatowski [5], who used the generalization of diBruno's formula given by Teixeira [7]; compare also Dederick [2].

The extension of (12) and (14) to the general case is purely a matter of notation, which should be sufficiently evident.

4. **Function of a function of a function.** For completeness, we note briefly another extension to a function of a function of a function, which contains the essence of a general extension. Take

$$F = f[g(h(x))]. \tag{15}$$

Then

$$\begin{aligned} F_n &= Y_n(fG_1, \cdots, fG_n), & f^i &\equiv f_i, \\ &= Y_n(\Gamma h_1, \cdots, \Gamma h_n), & \Gamma^i &\equiv \Gamma_i, \end{aligned} \tag{16}$$

with

$$\begin{aligned} g_n &= Y_n(gh_1, \cdots, gh_n), & g^i &\equiv g_i, \\ \Gamma_n &= Y_n(fg_1, \cdots, fg_n), & f^i &\equiv f_i. \end{aligned}$$

The last two expressions exemplify a kind of associative equality which permits writing the general result in a variety of ways.

Bibliography

1. E. T. Bell, *Exponential polynomials*, Ann. of Math. vol. 35 (1934) pp. 258–277.

2. L. S. Dederick, *Successive derivatives of a function of several functions*, Ann. of Math. vol. 27 (1925–26) pp. 385–394.

3. A. Dresden, *Derivatives of composite functions*, Amer. Math. Monthly vol. 50 (1943) pp. 9–12.

4. K. Menger, *Algebra of analysis*, Notre Dame Mathematical Lectures, No. 3, Notre Dame, Indiana, 1944.

5. I. Opatowski, *Combinatoric interpretation of a formula for the nth derivative of a function of a function*, Bull. Amer. Math. Soc. vol. 45 (1939) p. 944.

6. O. Schlömilch, *Compendium der höheren analysis*, s. 4, vol. 2, Braunschweig, 1879.

7. F. G. Teixeira, *Sur les dérivées d'ordre quelconque*, Giornale di Matematica di Battaglini vol. 18 (1880) p. 306.

8. H. S. Wall, *On the nth derivative of f(x)*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 395–397.

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