THE KLINE SPHERE CHARACTERIZATION PROBLEM

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The object of this paper is to give a solution to the following problem proposed by J. R. Kline: Is a nondegenerate, locally connected, compact continuum which is separated by each of its simple closed curves but by no pair of its points homeomorphic with the surface of a sphere? The answer is in the affirmative.

A solution to the Kline problem gives a characterization of a simple closed surface. Partial solutions of this problem have been made by Hall [1, 2][1] and Jones [3]. Other characterizations of a simple closed surface have been given by Kuratowski [4], Zippin [5, 6], Wilder [7] and Claytor [8]. Previous to the giving of these characterizations, Moore gave [9] two sets of axioms, each set of which characterized a set topologically equivalent to a plane.

DEFINITION. We say that M disrupts X from Y in D if there is an arc from X to Y in D but each such arc contains a point of M.

We shall make use of the following lemma.

LEMMA. *Suppose that space is locally connected and cannot be separated by the omission of any pair of its points, that the boundary of the connected domain D is equal to the sum of the mutually exclusive sets M, N and E, each of which contains a point which is accessible from D, and that D' is a connected subdomain of D such that no point of D either disrupts D' from $E+M$ in $D+E+M$ or disrupts D' from $E+N$ in $D+E+N$. Then there is an open arc from M to N in D that does not disrupt D' from E in $D+E$.*

PROOF. Consider the arc AB in $D+B$ from a point A of D' to a point B of E. Let W_1 be the set of all points P of AB such that there is an open arc from P to E in D that does not intersect some open arc from M to N in D. Assume that the first point R of AB in the order from A to B on the closure of W_1 does not belong to D'.

If R disrupts D' from E in $D+E$, there are an arc from D' to M in $D+M-R$ and an arc from D' to N in $D+N-R$. In the sum of these two arcs plus D' there is an open arc from M to N in D which does not intersect RB. This is contrary to the definition of R. Hence, R does not disrupt D' from E in $D+E$.

Let $A'B'$ be an arc in $D+B'-R$ from a point A' of D' to a point

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[1] Numbers in brackets refer to the references cited at the end of the paper.

B' of E. Since D' is connected and does not contain R, we may suppose that A' is A. Let X be the last point of $A'B'$ in the order from A' to B' on AR and let Y be a point of AB between R and B such that RY contains no point of XB'.

Choose a metric for $XB'+XY$. Let P_1 be a point of $W_1 \cdot XY$ such that no point of W_1 at a distance of more than 1 from P_1 is between P_1 and X on XY. Denote by V_1 the set of all points P of XB' such that there are in D an open arc from P to E and an open arc from XP_1 to E such that the sum of these two open arcs does not intersect some open arc from M to N in D. Let Q_1 be a point of V_1 such that no point of V_1 at a distance of more than 1 from Q_1 is between Q_1 and X on XB'.

Denote by W_2 the set of all points P of XY such that there are in D an open arc from P to E and an open arc from XQ_1 to E such that the sum of these open arcs does not intersect some open arc from M to N in D. Let P_2 be a point of $W_2 \cdot XP_1$ such that no point of W_2 at a distance of more than $1/2$ from P_2 is between P_2 and X on XY.

In general, let W_n (or V_n) be the set of all points P of XY (or XB') such that there are an open arc from P to E in D and an open arc from XQ_{n-1} (or XP_n) to E in D such that the sum of these open arcs does not intersect some open arc from M to N in D. Let P_n (or Q_n) be a point of W_n (or V_n) on XP_{n-1} (or XQ_{n-1}) such that no point of W_n (or V_n) at a distance of more than $1/n$ from P_n (or Q_n) is between P_n (or Q_n) and X on XY (or XB').

Let P_0 and Q_0 be the limit points of $P_1, P_2, \cdots$ and $Q_1, Q_2, \cdots$ respectively. We note that for no integer n is P_n between X and P_0 on XY or is Q_n between X and Q_0 on XB'.

Since no pair of points separates space, there is an arc from A to E in space that contains neither P_0 nor Q_0. Let $A''B''$ be a subset of this arc irreducible from $AX+XP_0+XQ_0$ to E. We shall suppose that A'' is a point of $AX+XP_0$ since the argument to follow may be adjusted to take care of the case where it belongs to $AX+XQ_0$.

There is an integer i such that AA'' contains no point of W_i and $A''B''$ intersects neither P_0P_i nor Q_0Q_i. There is an arc $M'N'$ in $D+M'+N'$ from a point M' of M to a point N' of N such that there are open arcs $\langle P'B_1\rangle$ and $\langle Q'B_2\rangle$ in $D-\langle M'N'\rangle$, where P' and Q' are points of P_0P_i and Q_0Q_i respectively and B_j ($j=1, 2$) is a point of E. Denote by P'' the first point of XP' in the order from X to P' on $P'B_1+Q'B_2$ and denote by Q'' the first point of XQ' in the order from X to Q' on $P'B_1+Q'B_2$. Let $P''B_3$ and $Q''B_4$ be arcs in $P'B_1+Q'B_2$ where B_3 and B_4 are points of B_1+B_2.

Denote the set $AX+XP''+XQ''$ by T. Since no point of AX is a point of W_i, $P''B_3+Q''B_4$ does not intersect AX. Then $P''B_3+Q''B_4$ intersects T only at $P''+Q''$.

Now $M'N'$ intersects T or else there is an open arc from A to E in $T+P''B_3$ and R is not as assumed. Also, $M'N'$ does not intersect T in only one point, for if it did, there would be an open arc from X to E in $T+P''B_3+Q''B_4$ which would not intersect $M'N'$. But X is not a point of W_i.

Let M'' and N'' be the first and last points of $M'N'$ in the order from M' to N' on T. Replace the part of $M'N'$ between M'' and N'' by an arc α from M'' to N'' in T. Now one of the points M'', N'' belongs to XP'', for if neither were a point of XP'', there would be an open arc from X to E in $XP''+P''B_3$ which would not intersect $M'M''+\alpha+N''N'$. This contradicts the fact that X is not a point of W_i. Also, one of M'', N'' is a point of XQ''. Since A'' is not a point of W_i, it is not between P'' and $M''+N''$ on XP''.

Let C be the first point of $A''B''$ in the order from A'' to B'' on $M+M'M''+N+N''N'+E+P''B_3+Q''B_4$. Now C is not a point of $E+P''B_3+Q''B_4$, for if it were, there would be an open arc from A'' to E in $A''C+P''B_3+Q''B_4$ that would not intersect $M'M''+\alpha+N''N'$. But A'' is not a point of W_i.

Assume that C is a point of $M+M'M''$. Consider an arc β from M to N in $A''C+M'M''+T+N''N'$ containing $A''C$. We note that β does not contain M''. Now A'' is not a point of XP'', for if it were, there would be an open arc from A'' to E in $P''B_3+Q''B_4+T$ which would not intersect β. Also, A'' is not a point of AX or else there would be an open arc from X to E in $P''B_3+Q''B_4+T$ which would not intersect β. But X is not a point of W_i. Hence, C is not a point of $M+M'M''$. Likewise, we find that C is not a point of $N+N''N'$.

This establishes the lemma. The assumption that R is not a point of D' leads to the contradiction that the point C of $M+M'M''+N+N''N'+E+P''B_3+Q''B_4$ belongs to neither $E+P''B_3+Q''B_4$, $M+M'M''$ nor $N+N''N'$.

THEOREM. *Suppose that S is a nondegenerate, compact continuous curve such that no pair of points but every simple closed curve of S separates it. Then S is topologically equivalent to the surface of a sphere.*

PROOF. Regard S as space. It is known [**6**, **10**] that S is a simple closed surface if no arc separates it. We shall show that no arc separates S.

Before giving the details of the proof, we shall briefly outline what we intend to do. On the assumption that some arc separates space, we shall get a finite collection H_1 of connected domains such that their sum does not separate space and such that the sum of any two nonintersecting elements of H_1 separates the sum of the elements of H_1. Collections H_2, H_3, $\cdots$ are defined which satisfy corresponding conditions and which are such that the closure of each element of H_{n+1} is a subset of the sum of the elements of H_n. See the figure. There

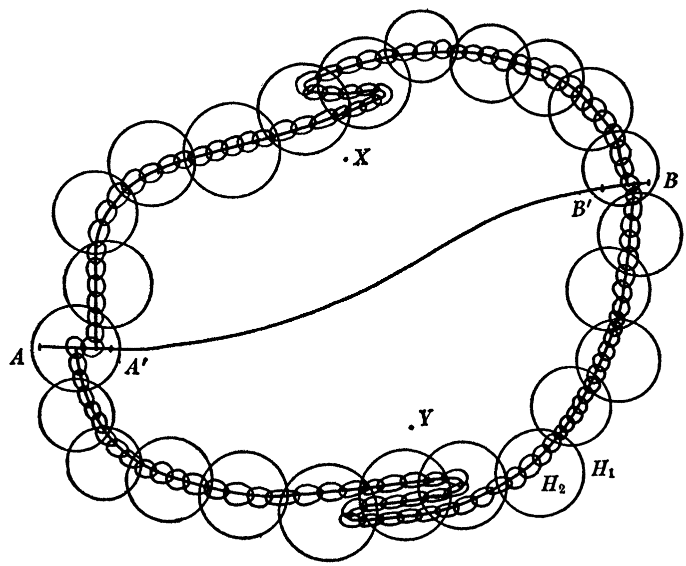

FIG. 1

are actually many more elements in H_1 and H_2 than are shown in the figure. The collections H_1, H_2, $\cdots$ are described in such a way that the common part of their sums is a simple closed curve not separating space. Hence, we shall show that the assumption that some arc separates space leads to the contradiction that some simple closed curve does not. We now consider the details of the proof.

Assume that an arc separates the point X from the point Y. Then there is an arc AB that separates X from Y such that no proper subarc of AB separates X from Y. Let D_X and D_Y be the complementary domains of AB containing X and Y respectively.

Description of collection H_1. Assume that a metric has been chosen for S and let ϵ_1 be a positive number less than one one-hundredth of the distance from A to B. We shall describe a collection H_1 of connected domains. The sum of the elements of H_1 will be denoted by H_1^*.

The collection H_1 of connected domains $h_{1,1}$, $h_{1,2}$, $\cdots$, $h_{1,t}$ $(t>100)$ will satisfy the following conditions:

(1) $h_{1,i}$ intersects $h_{1,j}$ only if i is equal to either $j-1$, j or $j+1$ $(h_{1,t+1}\equiv h_{1,1}$ and $h_{1,1-1}\equiv h_{1,t})$;

(2) $S-H_1^*$ is connected;

(3) some point of $S-H_1^*$ is accessible from $h_{1,i}$;

(4) the diameter of $h_{1,i}$ is less than ϵ_1;

(5) no connected subset of H_1^* that intersects $h_{1,i}$ and $h_{1,i+2}$ is of diameter less than $\epsilon_1/4$.

Denote by D_1, D_2, $\cdots$, D_n the elements of a finite collection of connected domains covering S such that the diameter of each is less than $\epsilon_1/100$. Suppose that each of the domains D_1, D_2, $\cdots$, D_j intersects the complement of D_X+D_Y, each of the domains D_{j+1}, D_{j+2}, $\cdots$, D_k is a subset of D_X and each of the domains D_{k+1}, D_{k+2}, $\cdots$, D_n is a subset of D_Y.

Let α_1, α_2, $\cdots$, α_j be a collection of arcs in the complement of D_X+D_Y+A+B such that α_i $(i=1, \cdots, j)$ intersects D_i and AB.

Let A' and B' be points of the arc AB such that AA' and $B'B$ are arcs that do not intersect $\alpha_1+\alpha_2+\cdots+\alpha_j$, each of the arcs AA' and $B'B$ is covered by an element of D_1, D_2, $\cdots$, D_n, some point of $A'B'$ is accessible from $D_Z(Z=X, Y)$ and if D_p is an element of D_1, D_2, $\cdots$, D_n in D_Z, then no point of D_Z either disrupts D_p from AB' in D_Z+AB' or disrupts D_p from $A'B$ in $D_Z+A'B$.

Considering $D_X, D_{j+1}, \overline{D}_X\cdot(AA'-A'), \overline{D}_X\cdot(B'B-B')$ and $\overline{D}_X\cdot A'B'$ as D, D', M, N and E of the preceding lemma, we find that there is an arc α_{j+1} from $A'B'$ to D_{j+1} in $D_X+A'B'$ that does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$. Let D' be a component of $D_X-D_X\cdot\alpha_{j+1}$ that contains an open arc from a point of $AA'-A'$ to a point of $B'B-B'$. If D_{j+2} is not a subset of D', let α_{j+2} be an arc in $D_X-D'+A'B'$ from $A'B'$ to a point of D_{j+2}. If D_{j+2} is a subset of D', we shall apply the lemma to get an arc α_{j+2} from D_{j+2} to $A'B'$ in $D_X+A'B'$ such that $\alpha_{j+1}+\alpha_{j+2}$ does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$. The procedure is described in the following paragraph.

Let R be a point of D'. Since no point of D_X disrupts D_{j+2} from AB' in D_X+AB', there is an arc β from D_{j+2} to AB' in $D_X+AB'-R$. A subarc of β in $D'+AB'+\alpha_{j+1}-R$ intersects D_{j+2} and $AB'+\alpha_{j+1}$. Hence, R does not disrupt D_{j+2} from $AB'+\alpha_{j+1}$ in $D'+\overline{D}'\cdot(AB'+\alpha_{j+1})$. Also, R does not disrupt D_{j+2} from $A'B+\alpha_{j+1}$ in $D'+\overline{D}'\cdot(A'B+\alpha_{j+1})$. Applying the lemma, we find that there is an arc from D_{j+2} to $A'B'+\alpha_{j+1}$ in $D'+A'B'+\alpha_{j+1}$ that does not disrupt $AA'-A'$ from $B'B-B'$ in $D'+AB-A'B'$. Then there is an arc α_{j+2} in $D_X+A'B'$

from D_{j+2} to $A'B'$ such that $\alpha_{j+1}+\alpha_{j+2}$ does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$.

Likewise, we find that there is an arc α_{j+3} from D_{j+3} to $A'B'$ in $D_X+A'B'$ such that $\alpha_{j+1}+\alpha_{j+2}+\alpha_{j+3}$ does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$. A continuation of this process gives that there are arcs $\alpha_{j+1}, \alpha_{j+2}, \cdots, \alpha_k$ in $D_X+A'B'$ whose sum does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$ and such that $\alpha_p(p=i+1, \cdots, k)$ intersects D_p and $A'B'$. Also, there are arcs $\alpha_{k+1}, \alpha_{k+2}, \cdots, \alpha_n$ in $D_Y+A'B'$ whose sum does not disrupt $AA'-A'$ from $B'B-B'$ in $D_Y+AB-A'B'$ and such that α_p $(p=k+1, \cdots, n)$ intersects D_p and $A'B'$.

Let G be the collection of all domains g such that g is a component of the common part of the complement of $AB+\alpha_1+\alpha_2+ \cdots +\alpha_n$ and some domain of $D_1, D_2, \cdots, D_n$. If P is a point of D_i, there is an arc in D_i from P to α_i. Hence, if g is an element of G, some point of $AB+\alpha_1+\alpha_2+ \cdots +\alpha_n$ is accessible from g.

Let g_A and g_B be domains of diameters less than $\epsilon_1/100$ that cover $AA'-A'$ and $B'B-B'$ respectively but no point of $A'B'+\alpha_1+\alpha_2 + \cdots +\alpha_n$. There exists a finite collection G_X of domains of G such that this collection but no collection of fewer elements of G satisfies the condition that the sum of the elements of G_X is a connected subset of D_X and intersects both g_A and g_B. Denote the elements of G_X by $g_2, g_3, \cdots, g_r$, where g_A intersects g_2, g_i $(i=2, \cdots, r-1)$ intersects g_{i+1} and g_r intersects g_B. Denote g_A by g_1 and g_B by g_{r+1}.

Also, there exists a finite collection G_Y of domains of G such that this collection but no collection of fewer elements of G satisfies the condition that the sum of the elements of G_Y is a connected subset of D_Y and intersects both g_A and g_B. Denote the elements of G_Y by $g_{r+2}, g_{r+3}, \cdots, g_s$ where g $(i_i=r+1, \cdots, s-1)$ intersects g_{i+1} and g_s intersects g_1.

Let E denote $g_1+g_2+ \cdots +g_s$ plus all points that it separates from $A'B'$. Each component of the common part of E and an element of G intersects an element of $g_1, g_2, \cdots, g_s$. However, it is to be noted that no such component intersects two g_i's that do not belong to a consecutive set of three domains of $g_1, g_2, \cdots, g_s, g_1, g_2$. Denote by g_i' the sum of g_i and all such components that intersect g_i. We note that g_i' is a diameter less than $\epsilon_1/33$.

If three is a factor of s, denote the sum of the first three elements of $g_1', g_2', \cdots, g_s'$ by h_1, the sum of the next three elements by $h_2, \cdots$ and the sum of the last three elements by h_v. If three is a factor of $s+1$, then $h_1, h_2, \cdots$ and h_v are defined as before except that h_v is the sum of the last four elements of $g_1', g_2', \cdots, g_s'$ instead of the last three. If three is a factor of $s+2$, each of h_{v-1} and h_v is the sum of four elements of $g_1', g_2', \cdots, g_s'$. Since each h_i contains either g_A, g_B or an element g of G which does not intersect g_A+g_B and since a point of $AB+\alpha_1+\alpha_2+\cdots+\alpha_n$ is accessible from g, then a point of $A'B'+\alpha_1+\alpha_2+\cdots+\alpha_n$ is accessible from h_i. Now h_i is of diameter less than $\epsilon_1/8$ and the collection $h_1, h_2, \cdots, h_v$ satisfies conditions analogous to conditions (1), (2) and (3) to be satisfied by $h_{1,1}, h_{1,2}, \cdots, h_{1,t}$.

Let $h_{1,1}$ be the sum of $h_1, h_2, \cdots, h_n$ where some connected subset of $h_1+h_2+\cdots+h_v$ of diameter less than $\epsilon_1/4$ intersects $h_1, h_2, \cdots$ and h_n but no such subset intersects both h_1 and h_{n+1}; let $h_{1,2}$ be the sum of $h_{n+1}, h_{n+2}, \cdots, h_m$ where some connected subset of $h_1+h_2+\cdots+h_v$ of diameter less than $\epsilon_1/4$ intersects $h_{n+1}, h_{n+2}, \cdots$ and h_m but no such subset intersects both h_{n+1} and h_{m+1}; $\cdots$; and let $h_{1,t}$ be the sum of $h_{p+1}, h_{p+2}, \cdots, h_v$ where some connected subset of $h_1+h_2+\cdots+h_v$ of diameter less than $3\epsilon_1/4$ intersects $h_{p+1}, h_{p+2}, \cdots$ and h_v but no subset of $h_1+h_2+\cdots+h_v$ of diameter less than $\epsilon_1/4$ intersects both h_{p+1} and h_1. We use $3\epsilon_1/4$ in the last case instead of $\epsilon_1/4$ in order to insure that no connected set in $h_1+h_2+\cdots+h_v$ of diameter less than $\epsilon_1/4$ intersects both h_{p+1} and h_1. The collection H_1 satisfies conditions (1), (2), (3), (4) and (5).

Description of collection H_2. Choose a positive number ϵ_2 less than one one-hundredth of the diameter of any connected set in H_1^* that intersects $h_{1,i}$ and $h_{1,i+2}$. We shall describe a collection H_2 of connected domains $h_{2,1}, h_{2,2}, \cdots, h_{2,s}$ such that:

(1) $h_{2,i}$ intersects $h_{2,j}$ only if i is equal to either $j-1$, j or $j+1$ ($h_{2,s+1}\equiv h_{2,1}$ and $h_{2,1-1}\equiv h_{2,s}$);

(2) $S-H_2^*$ is connected;

(3) some point of $S-H_2^*$ is accessible from $h_{2,i}$;

(4) the diameter of $h_{2,i}$ is less than ϵ_2;

(5) no connected subset of H_2^* that intersects $h_{2,i}$ and $h_{2,i+2}$ is of diameter less than $\epsilon_2/4$;

(6) each $h_{1,i}$ contains 98 consecutive elements of $h_{2,1}, h_{2,2}, \cdots, h_{2,s}$;

(7) if $H(n;\, i,\, j)$ denotes $h_{n,i-100}+\cdots+h_{n,i}+\cdots+h_{n,j}+\cdots+h_{n,j+100}$ and $h_{1,i}$ and $h_{1,j}$ intersect $h_{2,m}$ and $h_{2,k}$ respectively, then either $H(1;i,j)$ covers the closure of $H(2;m,k)$ and $H(1;j,i)$ covers the closure of $H(2;k,m)$ or $H(1;i,j)$ covers the closure of $H(2;k,m)$ and $H(1;j,i)$ covers the closure of $H(2;m,k)$.

Denote by C the component that contains $h_{1,5}+h_{1,6}$ of the common part of $h_{1,2}+h_{1,3}+\cdots+h_{1,9}$ and the complement of the closure of $h_{1,1}+h_{1,10}$. We shall show that if P is a point of $h_{1,5}+h_{1,6}$ and R is a point of $C-P$, then R does not disrupt P from $S-H_1^*$ in $S-H_1^*+C$. Let PQ be an arc in $S-R$ from P to a point Q of $S-H_1^*$. Let Q' be the first point of PQ in the order from P to Q on $S-C$. If PQ' intersects $h_{1,3}$, then there is an arc from $PQ'-Q'$ to $S-H_1^*$ in $S-H_1^*+h_{1,3}$ because a point of $S-H_1^*$ is accessible from $h_{1,3}$. Also, if PQ' intersects $h_{1,8}$, R does not disrupt P from $S-H_1^*$ in $S-H_1^*+C$. If PQ' intersects neither $h_{1,3}$ nor $h_{1,8}$, then Q' is a point of $S-H_1^*$. This demonstrates that R does not disrupt P from $S-H_1^*$ in $S-H_1^*+C$.

Let G be a finite collection of connected domains covering $h_{1,5}+h_{1,6}$ such that each intersects $h_{1,5}+h_{1,6}$ and is of diameter less than $\epsilon_2/1200$. No point of C disrupts an element of G from $S-H_1^*$ in $S-H_1^*+C$. Repeated applications of the preceding lemma give that there is a continuum K in $S-H_1^*+C$ that intersects $A'B'$ and each element of G but does not disrupt $h_{1,1}$ from $h_{1,10}$ in $h_{1,1}+h_{1,2}+\cdots+h_{1,10}$. Let G' be the set of all domains g' such that g' is either the common part of $h_{1,2}+h_{1,3}+\cdots+h_{1,9}$, the complement of K and an element of G, or the common part of the complement of K and h_i for i equal to 2, 3, 4, 7, 8 or 9.

There exists a finite collection G'' of elements of G' such that the sum of the elements of G'' is a connected domain intersecting $h_{1,1}$ and $h_{1,10}$ but the sum of no subcollection of G' having fewer elements than G'' is a connected domain intersecting $h_{1,1}$ and $h_{1,10}$. Assume that g_1 of G'' intersects $h_{1,1}$, g_i $(i=1,\cdots,r-1)$ intersects g_{i+1} and g_r intersects $h_{1,10}$.

There exists a collection $g_1', g_2',\cdots,g_r'$ of connected domains such that g_1' intersects $h_{1,1}$, g_i' intersects g_{i+1}', g_r' intersects $h_{1,10}$ and the closure of g_k' $(k=1,\cdots,r)$ is a subset of g_k.

Let E denote $h_{1,1}+g_1'+\cdots+g_r'+h_{1,10}+\cdots+h_{1,t}$ plus all points that it separates from $A'B'$. Each component of the common part of E and an element of G' intersects one of the domains $h_{1,1}$, $g_1',\cdots,g_r'$, $h_{1,10}$ but no such component intersects two of these domains that do not belong to a consecutive set of three of these domains. Add such components to the ones of $h_{1,1}$, g_i', $h_{1,10}$ that they intersect to form the sets h_1, g_i'', h_{10}. We note that the diameter of each g_i'' not intersecting $h_{1,2}+h_{1,3}+h_{1,4}+h_{1,7}+h_{1,8}+h_{1,9}$ is less than $\epsilon_2/400$.

Consecutive elements of $g_1'', g_2'',\cdots,g_r''$ may be combined by threes and fours in a manner previously described so as to get a collection $g_{1,1}, g_{1,2},\cdots,g_{1,u}$ such that the collection $h_1, g_{1,1},\cdots,g_{1,u}, h_{10}, h_{1,11},\cdots,h_{1,t}$ satisfies conditions analogous to conditions (1), (2) and (3) to be satisfied by $h_{2,1}, h_{2,2},\cdots,h_{2,s}$. We note that the closure of $g_{1,i}$ $(2\leqq i<u)$ is a subset of $h_{1,1}+h_{1,2}+\cdots+h_{1,10}$.

In a manner similar to that in which $h_{1,1}+h_{1,2}+ \cdots +h_{1,10}$ was replaced by $h_1+g_{1,1}+ \cdots +g_{1,u}+h_{10}$, we replace $h_{1,11}+ \cdots +h_{1,20}$ by $h_{11}+g_{11,1}+ \cdots +g_{11,v}+h_{20}, \cdots$ and $h_{1,t-m}+ \cdots +h_t$ $(9 \leqq m \leqq 18)$ by $h_{t-m}+g_{t-m,1}+ \cdots +g_{t-m,w}+h_t$.

Let $g_{i,0}$ be the fourth element of $g_{i,1}, g_{i,2}, \cdots, g_{i,n}$ which follows all of these elements that intersect $h_{1,i+3}$. We note that $g_{i,0}$, the three domains immediately preceding $g_{i,0}$, and the three domains immediately following $g_{i,0}$ are each a subset of $h_{1,i+4}$ of diameter less than $\epsilon_2/100$.

In the manner described above, replace $g_{1,0}+ \cdots +h_{10}+h_{11}+ \cdots +g_{11,0}$ by $g_{1,0}'+h_{2,2}'+ \cdots +h_{2,r}'+g_{11,0}'$; replace $g_{11,0}'+ \cdots +g_{21,0}$ by $h_{2,r+1}'+ \cdots +h_{2,v}'+g_{21,0}'; \cdots$ and replace $g_{t-m,0}'+ \cdots +g_{1,0}'$ by $h_{2,n}'+ \cdots +h_{2,u}'+h_{2,1}'$. The closure of $h_{2,2}'+ \cdots +h_{2,r}'$ is a subset of $h_{1,4}+ \cdots +h_{1,20}, \cdots$ and the closure of $h_{2,n}'+ \cdots +h_{2,u}'+h_{2,1}'$ is a subset of $h_{1,t-m+3}+ \cdots +h_{1,10}$. Consecutive elements of $h_{2,1}', h_{2,2}', \cdots, h_{2,u}'$ may be combined in a manner previously described so as to form a collection H_2 of connected domains $h_{2,1}, h_{2,2}, \cdots, h_{2,s}$ satisfying conditions (1), (2), (3), (4), (5), (6) and (7).

Description of simple closed curve J. For each positive integer i greater than one, we define a collection H_i of connected domains $h_{i,1}, h_{i,2}, \cdots, h_{i,n}$ satisfying conditions analogous to those satisfied by H_2 where ϵ_i is a positive number less than one one-hundredth the diameter of any connected set in H_{i-1}^* intersecting $h_{i-1,j}$ and $h_{i-1,j+2}$. We shall show that the common part J of $H_1^*, H_2^*, \cdots$ is a simple closed curve that does not separate S.

As the closure of H_{i+1}^* is a connected subset of H_i^* (condition 7) and as each $h_{i,j}$ contains an element of H_{i+1} (condition 6), then J is a nondegenerate continuum. This continuum does not separate space because the complement of each H_i^* is connected.

To show that J is a simple closed curve, we shall show that any pair of points P, Q of J separates it. Suppose that h_{i,P_i} and h_{i,Q_i} are elements of $h_{i,1}, \cdots, h_{i,n}$ that contain P and Q respectively. For convenience in notation, we shall assume that it is $H(i; P_i, Q_i)$ that covers the closure of $H(i+1; P_{i+1}, Q_{i+1})$ and that it is $H(i; Q_i, P_i)$ that covers the closure of $H(i+1; Q_{i+1}, P_{i+1})$. If J_{PQ} is the common part of $H(1; P_1, Q_1), H(2; P_2, Q_2), \cdots$, we find that J is the sum of two continua J_{PQ} and J_{QP} which have only P and Q in common.

Hence, the assumption that an arc separates S leads to the conclusion that some simple closed curve does not.

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