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A REMARK ON DENSITY CHARACTERS

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Let X be an arbitrary topological space satisfying the T_0-separation axiom [**1**, Chap. 1, §4, p. 58].[2] We recall the following definition [**3**, p. 329].

DEFINITION 1. *The least cardinal number of a dense subset of the space X is said to be the density character of X. It is denoted by the symbol $\Xi(X)$.*

We denote the cardinal number of a set A by $|A|$.

Pospíšil has pointed out [**4**] that if X is a Hausdorff space, then

$$|X| \leqq 2^{2^{\Xi(X)}}. \tag{1}$$

This inequality is easily established. Let D be a dense subset of the Hausdorff space X such that $|D| = \Xi(X)$. For an arbitrary point $p \in X$ and an arbitrary complete neighborhood system $\mathcal{V}_p$ at p, let $\mathcal{D}_p$ be the family of all sets $U \cap D$, where $U \in \mathcal{V}_p$. Thus to every point of X, a certain family of subsets of D is assigned. Since X is a Hausdorff space, $\mathcal{D}_p \neq \mathcal{D}_q$ whenever $p \neq q$, and the correspondence assigning each point p to the family $\mathcal{D}_p$ is one-to-one. Since X is in one-to-one correspondence with a sub-hierarchy of the hierarchy of all families of subsets of D, the inequality (1) follows.

It may be remarked in passing that the inequality (1) does not obtain for all T_1-spaces. Let $\mathfrak{m}$ be a cardinal number greater than $2^{\mathfrak{c}}$, where $\mathfrak{c} = 2^{\aleph_0}$. Let Z be a T_1-space of cardinal number $\mathfrak{m}$ and with the property that the only closed proper subsets of Z are finite or

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[2] Numbers in brackets refer to the Bibliography at the end of the paper.

void. Then it is obvious that $\Xi(Z)=\aleph_0$, and that (1) does not obtain for the space Z.

For some Hausdorff spaces, the equality

$$(2) \qquad |X| = 2^{2^{\Xi(X)}}$$

obtains. Pospíšil [4] has constructed a large family of such Hausdorff spaces, and has shown [5] that the Stone-Čech β for any discrete infinite space satisfies it as well. It is the purpose of this note to exhibit another class of Hausdorff spaces for which (2) holds.

THEOREM. *Let* Λ *be an index class such that* $|\Lambda|=2^{\mathfrak{m}}$ *where* $\mathfrak{m}$ *is an infinite cardinal number. Let* $\{X_\lambda\}_{\lambda\in\Lambda}$ *be a family of Hausdorff spaces such that* $|X_\lambda|\geqq 2$ *and* $\Xi(X_\lambda)\leqq\mathfrak{m}$ *for all* $\lambda\in\Lambda$. *Then* $|\mathfrak{P}_{\lambda\in\Lambda}X_\lambda|=2^{2^{\mathfrak{m}}}$ *and* $\Xi(\mathfrak{P}_{\lambda\in\Lambda}X_\lambda)=\mathfrak{m}$.

PROOF. We first consider the set Λ as a topological space itself. Clearly, it may be put into one-to-one correspondence with the Cartesian product $\mathfrak{P}_{\mu\in M}A_\mu$, where each A_μ is a Hausdorff space containing exactly two points and the index class M has cardinal number $\mathfrak{m}$. As is well known, this Cartesian product is a bicompact Hausdorff space with cardinal number $2^{\mathfrak{m}}$ and a basis of open sets with cardinal number $\mathfrak{m}$. We may consequently regard Λ as being a Hausdorff space with a basis $\mathcal{B}$ of open sets such that $|\mathcal{B}|=\mathfrak{m}$.

Let $y^0=\{q_\lambda^0\}$ be a fixed point in the space $P_{\lambda\in\Lambda}X_\lambda$. Let D_λ be a dense subset in X_λ such that $|D_\lambda|=\Xi(X_\lambda)\leqq\mathfrak{m}$. If α_0 is the least ordinal number with corresponding cardinal number $\mathfrak{m}$, then each set D_λ can be so well ordered that

$$D_\lambda = \{\overset{1}{p_\lambda}, \overset{2}{p_\lambda}, \overset{3}{p_\lambda}, \cdots, \overset{\alpha}{p_\lambda}, \cdots\}, \qquad \alpha<\alpha_0.$$

If $|D_\lambda|<\mathfrak{m}$, then the elements $p_\lambda{}^\alpha$ may be all taken identical from a certain point on. Of course, if $|D_\lambda|=\mathfrak{m}$, no repetitions need occur.

Let $\{\Lambda_1,\cdots,\Lambda_n\}$ be an arbitrary family of disjoint sets in $\mathcal{B}$, and let $\{\alpha_1,\cdots,\alpha_n\}$ be arbitrary ordinal numbers all less than α_0. Let $x(\Lambda_1,\cdots,\Lambda_n;\alpha_1,\cdots,\alpha_n)=\{r_\lambda\}$ be the point in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$ such that $r_\lambda=p_\lambda{}^{\alpha_i}$ for all $\lambda\in\Lambda_i$, $i=1, 2, 3, \cdots, n$, and $r_\lambda=q_\lambda{}^0$ for $\lambda\in\Lambda\cap(\sum_{i=1}^{n}\Lambda_i)'$. Let W be the set of all points $x(\Lambda_1,\cdots,\Lambda_n;$ $\alpha_1,\cdots,\alpha_n)$ as $\{\Lambda_1,\cdots,\Lambda_n\}$ and $\{\alpha_1,\cdots,\alpha_n\}$ assume all possible values. It is clear that $|W|=\sum_{n=1}^{\infty}\mathfrak{m}^n\cdot\mathfrak{m}^n=\aleph_0\cdot\mathfrak{m}=\mathfrak{m}$. Furthermore, W is dense in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$. Let G be an arbitrary non-void open set in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$. By the definition of open sets and neighborhoods in a Cartesian product (see, for example, [2, pp. 829–830]), there exist a finite subset $\{\lambda_1,\cdots,\lambda_m\}$ of Λ and sets $U_{\lambda_1}\cdots, U_{\lambda_m}$, where U_{λ_i} is an open set in X_{λ_i}, with the property that G contains all points $\{s_\lambda\}$

of $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$ such that $s_{\lambda_i}\in U_{\lambda_i}$ for $i=1, 2, 3, \cdots, m$. The sets D_λ being dense in the spaces X_λ, there is a point $p_{\lambda_i}^{\alpha_i}\in D_{\lambda_i}$ such that $p_{\lambda_i}^{\alpha_i}\in U_{\lambda_i}$ ($i=1, 2, 3, \cdots, m$). Since Λ is a Hausdorff space under the topology defined by $\mathcal{B}$, there are sets $\Lambda_1, \cdots, \Lambda_m$ in $\mathcal{B}$ such that $\Lambda_i \cap \Lambda_j = 0$ for $i\neq j$ and such that $\lambda_i\in\Lambda_i$ for all $i=1, 2, 3, \cdots, m$. It is obvious that the point $x(\Lambda_1, \cdots, \Lambda_m; \alpha_1, \cdots, \alpha_m)$ is in the set $W\cap G$. Having a nonvoid intersection with an arbitrary nonvoid open set in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$, W is dense in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$.

It follows from the definition of $\Xi(\mathfrak{P}_{\lambda\in\Lambda}X_\lambda)$ and the equality $|W| = \mathfrak{m}$ that $\Xi(\mathfrak{P}_{\lambda\in\Lambda}X_\lambda) \leqq \mathfrak{m}$. On the other hand, we have

$$|\mathfrak{P}_{\lambda\in\Lambda}X_\lambda| \geqq \mathfrak{m}^{|\Lambda|} = 2^{2^{\mathfrak{m}}}. \tag{3}$$

Hence, by virtue of the inequality (1), it follows that

$$|\mathfrak{P}_{\lambda\in\Lambda}X_\lambda| = 2^{2^{\mathfrak{m}}} \tag{4}$$

and

$$\Xi(\mathfrak{P}_{\lambda\in\Lambda}X_\lambda) = \mathfrak{m}. \tag{5}$$

This completes the proof.

For a result similar to this, see [6].

The foregoing theorem, applied to various well known spaces, yields curious results.

COROLLARY 1. *The space of all real-valued functions of a real variable with the Cartesian product topology contains a countable dense subset.*

COROLLARY 2. *The space of all characteristic functions defined on a set of cardinal number $2^{\aleph_0}$ contains a countable dense subset under the Cartesian product topology.*

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