ALMOST PERIODICITY, EQUI-CONTINUITY AND TOTAL BOUNDEDNESS

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Let X be a uniform space; that is to say, let X be a space provided with a system of indexed neighborhoods $U_\alpha(x)$ ($x \in X$, $\alpha =$ index), subject to the conditions (A. Weil): (1) If $x \in X$ and if α is an index, then $x \in U_\alpha(x)$; (2) If α and β are indices, then there exists an index γ such that $x \in X$ implies $U_\gamma(x) \subset U_\alpha(x) \cap U_\beta(x)$; (3) If α is an index, then there exists an index β such that $x, y, z \in X$ with $x, y \in U_\beta(z)$ implies $x \in U_\alpha(y)$. Let T be a topological group with identity σ and let f be a transformation of $X \times T$ into X. We agree to write $f^t(x)$ or $f_x(t)$ in place of $f(x, t)$ ($x \in X$, $t \in T$), whenever we wish. Furthermore, let f define a transformation group; that is to say, suppose $f^\sigma(x) = x$ and $f^s f^t(x) = f^{ts}(x)$ ($x \in X$; $t, s \in T$). We impose continuity conditions on f as the needs arise.

A subset E of T is said to be *relatively dense* provided there exists a compact set A in T such that each left translate of A intersects E. A point x of X is called *almost periodic* provided that if U is a neighborhood of x, then there exists a relatively dense set E in T for which $f(x, E) \subset U$. We observe that the notion of almost periodic point depends on the topology in T, the strongest type of almost periodicity occurring when T is provided with the discrete topology. It is easily proved that a set E in T is relatively dense if and only if there exists a compact set B in T such that $T = EB$. The set $f(x, T)$ is called the *orbit* of the point x.

THEOREM 1. *If the family* $[f^t \mid t \in T]$ *is equi-continuous at* x, *if* f_x *is continuous on* T, *and if* x *is almost periodic, then the orbit of* x *is totally bounded. Conversely, if the family* $[f^t \mid t \in T]$ *is equi-uniformly continuous and if the orbit of* x *is totally bounded, then* x *is almost periodic.*

PROOF. Suppose the hypotheses of the first statement hold. Let α be an index. There exists an index β such that the β-neighborhood of each compact set in X is contained in the union of finitely many α-neighborhoods. By hypothesis we can find an index γ such that $f^t(U_\gamma(x)) \subset U_\beta(f^t(x))$ ($t \in T$). There are sets E and A in T such that $T = EA$, A is compact, and $f(x, E) \subset U_\gamma(x)$. Hence, $f(x, T) \subset f(U_\gamma(x), A) \subset U_\beta(f(x, A))$. Since $f(x, A)$ is compact, $U_\beta(f(x, A))$ is contained in

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the union of finitely many α-neighborhoods. Thus the orbit of x is totally bounded.

Suppose the hypotheses of the second statement hold. Let α be an index. By hypothesis there exists an index β such that $f^t(U_\beta(y)) \subset U_\alpha(f^t(y))$ $(y \in X,\ t \in T)$. Choose finitely many elements $t_1, \cdots, t_n$ of T so that $f(x, T) \subset \bigcup_{i=1}^n V_i$, where $V_i = U_\beta(f(x, t_i))$. If $t \in T$, then for some i, $f(x, t) \in V_i$ whence $f(x, tt_i^{-1}) \in U_\alpha(x)$. Hence, x is almost periodic.

COROLLARY 1. *If the family $[f^t \mid t \in T]$ is equi-uniformly continuous, if f_x is continuous on T, and if x is almost periodic, then x is almost periodic with respect to the discrete topology in T.*

COROLLARY 2 (BOCHNER'S THEOREM). *If $x(\tau)$, $-\infty < \tau < +\infty$, is a complex-valued continuous function, then $x(\tau)$ is an almost periodic function in the sense of Bohr if and only if each sequence of translates of $x(\tau)$ contains a uniformly convergent subsequence.*

PROOF. Define Y to be the collection of all complex-valued continuous functions $y(\tau)$, $-\infty < \tau < +\infty$, and define indexed neighborhoods in Y so that $y \in U_n(y_0)$ if and only if $|y(\tau) - y_0(\tau)| < 1/n$ $(-\infty < \tau < +\infty)$ where the index n is a positive integer. Construct a transformation group g in Y by translation of the functions in Y, taking T to be the additive group of reals with its natural topology. Now x is an almost periodic point if and only if $x(\tau)$ is an almost periodic function. Clearly, the family $[g^t \mid t \in T]$ is equi-uniformly continuous. Also if $x(\tau)$ is an almost periodic function, then $x(\tau)$ is uniformly continuous whence g_x is continuous on T. Hence, by Theorem 1, x is an almost periodic point if and only if the orbit of x is totally bounded. The conclusion follows.

We point out that A. Weil [2, pp. 130–133][1] has essentially taken Theorem 1 as the definition of an almost periodic point with respect to a transformation group.

If X is an arbitrary set, if Y is a uniform space, and if Φ is a nonvacuous collection of functions on X to Y, then we consider Φ to be a uniform space in the following manner: If α is an index belonging to Y and if $\phi \in \Phi$, then the α-neighborhood $U_\alpha(\phi)$ of ϕ is taken to be the set of all elements ψ of Φ such that $\psi(x) \in U_\alpha(\phi(x))$ for every element x of X.

The following lemma will be recognized as a generalization of Ascoli's theorem and its converse.

LEMMA 1. *Let X and Y be uniform spaces and let Φ be a nonvacuous*

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

collection of functions on X to Y. If X and Y are totally bounded and if the family Φ is equi-uniformly continuous, then the space Φ is totally bounded. Conversely, if the individual functions in Φ are uniformly continuous and if the space Φ is totally bounded, then the family Φ is equi-uniformly continuous.

PROOF. Suppose X and Y are totally bounded and Φ is equi-uniformly continuous. Let α be an index belonging to Φ and, hence, to Y. Choose an index β belonging to Y so that $a, b, c \in Y$ with $a, b \in U_\beta(c)$ implies $a \in U_\alpha(b)$. We can find an index γ belonging to Y and finitely many points $y_1, \cdots, y_n$ of Y so that $y \in Y$ implies $U_\gamma(y) \subset U_\beta(y_j)$ for some integer j, $1 \leqq j \leqq n$. There exists an index δ belonging to X such that $\phi(U_\delta(x)) \subset U_\gamma(\phi(x))$, $(\phi \in \Phi, x \in X)$. Choose finitely many points $x_1, \cdots, x_m$ of X so that $X = \bigcup_{i=1}^m U_\delta(x_i)$. Hence if $\phi \in \Phi$, then to each integer i, $1 \leqq i \leqq m$, there corresponds at least one integer j, $1 \leqq j \leqq n$, such that $\phi(U_\delta(x_i)) \subset U_\beta(y_j)$. Letting I, J denote the first m, n positive integers, each element ϕ of Φ defines in the obvious manner a class $C(\phi)$ of transformations of I into J. Choose finitely many elements $\phi_1, \cdots, \phi_r$ of Φ so that the class $\bigcup_{k=1}^r C(\phi_k)$ is maximal. It follows that $\Phi = \bigcup_{k=1}^r U_\alpha(\phi_k)$. Hence, Φ is totally bounded.

Suppose now that the functions in Φ are uniformly continuous and Φ is totally bounded. Let α be an index belonging to Y. There exists an index γ belonging to Y so that $a \in U_\gamma(c)$, $b \in U_\gamma(d)$ and $c \in U_\gamma(d)$ implies $a \in U_\alpha(b)$. Choose finitely many elements $\phi_1, \cdots, \phi_n$ of Φ so that $\Phi = \bigcup_{i=1}^n U_\gamma(\phi_i)$. Select indices $\beta(i)$ $(i = 1, \cdots, n)$ belonging to X which have the property that $\phi_i(U_{\beta(i)}(x)) \subset U_\gamma(\phi_i(x))$ $(x \in X;$ $i = 1, \cdots, n)$. There exists an index β belonging to X for which $U_\beta(x) \subset \bigcap_{i=1}^n U_{\beta(i)}(x)$ $(x \in X)$. We conclude that $\phi(U_\beta(x)) \subset U_\alpha(\phi(x))$ $(\phi \in \Phi, x \in X)$. Hence, Φ is equi-uniformly continuous and the proof is completed.

We say that the transformation group f is *almost periodic* provided that if α is an index, then there exists a relatively dense set E in T for which $x \in X$ implies $f(x, E) \subset U_\alpha(x)$. It may be verified that in order for f to be almost periodic it is both necessary and sufficient that if α is an index, then there exists a compact set A in T such that to each element t in T there corresponds an element s in A for which $f^t(x) \in U_\alpha(f^s(x))$ $(x \in X)$. If T has the discrete topology, this characterization reduces essentially to Montgomery's definition [1, p. 323] of an almost periodic transformation group.

In the following theorem we denote $[f^t | t \in T]$ by G and, somewhat improperly, call G the transformation group.

THEOREM 2. *If X is compact and if f is continuous on $X \times T$, then* *the following statements are pairwise equivalent*: (1) *The transformation group G is almost periodic*; (2) *The family G is equi-uniformly continuous*; (3) *The space G is totally bounded.*

PROOF. By Lemma 1, (2) is equivalent to (3).

Suppose (1) is satisfied. Let α be an index. There exists an index γ such that $a\in U_\gamma(c)$, $b\in U_\gamma(d)$ and $c\in U_\gamma(d)$ implies $a\in U_\alpha(b)$. It is possible to find a compact set A in T such that if $t\in T$, then $f^t(x)\in U_\gamma(f^s(x))$ $(x\in X)$, for some element s in A. Since f is uniformly continuous on $X\times A$, we can choose an index β for which $f^s(U_\beta(x))\subset U_\gamma(f^s(x))$ $(x\in X,\ s\in A)$. It follows that $f^t(U_\beta(x))\subset U_\alpha(f^t(x))$ $(x\in X,\ t\in T)$. Hence, (2) is satisfied.

Suppose (2) is satisfied. Let α be an index. There exists an index β such that $f^t(U_\beta(x))\subset U_\alpha(f^t(x))$ $(x\in X,\ t\in T)$. Since G is totally bounded, we can select finitely many elements $t_1, \cdots, t_n$ in T so that $G=\bigcup_{i=1}^n U_\beta(f^{t_i})$. Let t be an element of T. Then for some i, $f(x, t)\in U_\beta(f(x, t_i))$ $(x\in X)$, whence $f(x, tt_i^{-1})\in U_\alpha(x)$ $(x\in X)$. Thus (1) is satisfied and the proof is completed.

COROLLARY 3. *If X is compact, if f is continuous on $X\times T$, and if f is almost periodic, then f is almost periodic with respect to the discrete topology in T.*

COROLLARY 4 (SHARPENED DIRICHLET-KRONECKER THEOREM). *If $t, a_1, \cdots, a_k$ are nonzero real numbers and if ϵ is a positive number, then there exists a relatively dense set N of integers such that $n\in N$ implies the existence of integers $m_1, \cdots, m_k$ for which $|nt-m_ia_i|<\epsilon$ $(i=1, \cdots, k)$.*

PROOF. Let $X_1, \cdots, X_k$ denote pairwise disjoint circle boundaries in the plane with circumferences $|a_1|, \cdots, |a_k|$. Take $X=\bigcup_{i=1}^k X_i$ with the natural uniformity. Define the uniformity-preserving homeomorphism ϕ of X onto X by rotating each circle X_i through arc length t. The transformation group generated by the integral powers of ϕ satisfies (2) of Theorem 2 and, hence, is almost periodic. The conclusion follows from the definition of almost periodicity.

BIBLIOGRAPHY

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