ON THE NUMBER OF 1-1 DIRECTLY CONFORMAL MAPS WHICH A MULTIPLY-CONNECTED PLANE REGION OF FINITE CONNECTIVITY p (>2) ADMITS ONTO ITSELF

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1. **Introduction.** It is well known[1] that a plane multiply-connected region G of finite connectivity greater than two admits only a finite number of 1-1 directly conformal maps *onto* itself (such maps will be termed henceforth *conformal automorphisms* of G); in fact, if G is of connectivity $p(>2)$, then the number of conformal automorphisms of G can in no case exceed $p(p-1)(p-2)$. The object of the present note is to determine the *best* upper bound, $N(p)$, for the number of conformal automorphisms of G as a *function of the connectivity* p. The basic theorems are:

THEOREM A. *The group of conformal automorphisms of a plane region of finite connectivity $p(>2)$ is isomorphic to one of the finite groups of linear fractional transformations of the extended plane onto itself.*

THEOREM B. *If $p(>2)$ is different from* 4, 6, 8, 12, 20, *then* $N(p)=2p$. *For the exceptional values of p, one has*

$$N(4) = 12, \qquad N(6) = N(8) = 24, \qquad N(12) = N(20) = 60.$$

The proofs of these theorems are based upon the following results:[2]

I. *An arbitrary plane region G of finite connectivity p admits a* 1-1 *directly conformal map onto a canonical plane region G^* whose boundary consists of points and complete circles (either possibly absent), in all p in number, and mutually disjoint.*

If $\mathcal{G}$ and $\mathcal{G}^*$ denote the groups of conformal automorphisms of G and G^* respectively, then $\mathcal{G}$ is isomorphic to $\mathcal{G}^*$. Hence for the purposes of the present problem it suffices to consider the canonical regions and their associated groups of conformal automorphisms.

II. *A conformal automorphism of a canonical region G^* admits an extension in definition throughout the extended complex plane as a linear fractional transformation.*

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[1] Cf. G. Julia, *Leçons sur la représentation conforme des aires multiplement connexes*, Paris, 1934. In particular, see pp. 68–69.

[2] Cf. Hurwitz-Courant, *Funktionentheorie*, Berlin, 1929. See pp. 512–520.

Hence the group $\mathcal{G}^*$ is in essence a *finite group* of linear fractional transformations of the extended plane onto itself. In the next section it will be shown that there exists a region Γ which lies in the extended complex plane, is bounded by p distinct points, and in addition

(i) contains G^* as a subregion,

(ii) remains *invariant* under the automorphisms of $\mathcal{G}^*$.

It will then follow that it suffices to consider the problem for regions bounded by p distinct points. The proof of Theorem B is thus reduced to the determination of the connectivities of regions Γ which are bounded by a finite set of points and which remain invariant under the members of a given finite group of linear fractional transformations of the extended complex plane onto itself.

2. **Reduction of problem to the case where the region is bounded by p distinct points.** We start then with a canonical region G^* whose boundary consists of p disjoint components which are either points or circles and the associated group of conformal automorphisms $\mathcal{G}^*$. Boundary components which consist of points will be unaltered. If there are circles present among the boundary components, say $\beta_1, \beta_2, \cdots, \beta_m$ $(1 \leqq m \leqq p)$, one proceeds as follows. Suppose β_k $(1 \leqq k \leqq m)$ is carried into itself by some transformation $S \in \mathcal{G}^*$ other than the identity. Note that S is an *elliptic* linear fractional transformation and hence possesses a unique fixed point ζ_k in the region G_k of the extended plane bounded by β_k which is exterior to G^*. Further any transformation of $\mathcal{G}^*$ which carries β_k into itself possesses ζ_k as a fixed point since the subgroup of $\mathcal{G}^*$ whose members preserve β_k is *cyclic*. For any such transformation (not the identity) ζ_k is the unique fixed point in g_k. In this case we replace β_k by the point ζ_k. If β_k is not carried into itself by any transformation of $\mathcal{G}^*$ other than the identity, then β_k and its images with respect to the transformations of $\mathcal{G}^*$ constitute a set of n disjoint circles, where n is the order of $\mathcal{G}^*$. These circles are permuted among themselves by the transformations of $\mathcal{G}^*$. To replace these circles by points, we select any one of them—say β_{k_0}—and fix a point η_{k_0} on β_{k_0} replacing thereby β_{k_0} by η_{k_0}. The image of β_{k_0} with respect to a transformation of $\mathcal{G}^*$ is to be replaced by the image of η_{k_0} with respect to the same transformation of $\mathcal{G}^*$. In this manner G^* is replaced by a region $\Gamma \supset G^*$ of connectivity p whose boundary consists of p distinct points. It is readily verified that Γ remains invariant with respect to the transformations of $\mathcal{G}^*$.

Hence, to determine

$$N(p) \equiv \max_{\{G\}} [\text{order } \mathcal{G}(G)],$$

where G is of connectivity p and $\mathcal{G}(G)$ is the group of conformal automorphisms of G, it suffices to consider regions G bounded by p distinct points.

3. **An observation.** It is to be observed that $N(p)$ is *bounded below by* $2p$. This follows from the fact that the region in the extended z-plane whose boundary consists of the points

$$z = e^{2\pi ik/p} \qquad (k = 0, 1, 2, \cdots, p-1)$$

is carried into itself by the dihedral group of order $2p$ generated by

$$S: z \mid 1/z, \qquad T: z \mid e^{2\pi i/p}z.$$

This fact will be significant for determining $N(p)$.

4. **Determination of** $N(p)$. Given a positive integer p, a finite group $\mathcal{G}$ of linear fractional transformations of the extended complex plane onto itself will be termed *admissible relative to* p, if there exists a region Γ which is bounded by p distinct points and remains invariant under the transformations of $\mathcal{G}$. Given $\mathcal{G}$, the integers p for which $\mathcal{G}$ is *admissible* are listed in the following table:[3]

TABLE 1

If $\mathcal{G}$ is isomorphic to	then p for which $\mathcal{G}$ is admissible are given by
Cyclic group of order n	$n\left[\frac{a}{n}+\frac{b}{1}\right]$ where $a=0, 1, 2$; $b=0, 1, 2, \cdots$; $a+b>0$
Dihedral group of order $2n$	$2n\left[\frac{a}{n}+\frac{b}{2}+\frac{c}{1}\right]$ where $a=0, 1$; $b=0, 1$; $c=0, 1, 2, \cdots$; $a+b+c>0$
Tetrahedral group	$12\left[\frac{a}{3}+\frac{b}{2}+\frac{c}{1}\right]$ where $a=0, 1, 2$; $b=0, 1$; $c=0, 1, 2, \cdots$; $a+b+c>0$
Octahedral group	$24\left[\frac{a}{4}+\frac{b}{3}+\frac{c}{2}+\frac{d}{1}\right]$ where $a=0, 1$; $b=0, 1$; $c=0, 1$; $d=0, 1, 2, \cdots$; $a+b+c+d>0$
Icosahedral group	$60\left[\frac{a}{5}+\frac{b}{3}+\frac{c}{2}+\frac{d}{1}\right]$ where $a=0, 1$; $b=0, 1$; $c=0, 1$; $d=0, 1, 2, \cdots$; $a+b+c+d>0$

[3] This table is readily verified on reference to the classical results of the theory of finite groups of linear fractional transformations.

Recall that all finite groups $\mathcal{G}$ of linear fractional transformations of the extended complex plane onto itself are considered in Table 1. Since $N(p) \geqq 2p$, it suffices to consider groups $\mathcal{G}$ admissible relative to $p(>2)$ *whose orders are at least* $2p$. These are readily determined from Table 1 and are given below together with their orders in Table 2.

TABLE 2

p	Groups $\mathcal{G}$ admissible relative to p and of order $\geqq 2p$	Order of $\mathcal{G}$
$\neq 4, 6, 8, 12, 20, 30$	Dihedral	$2p$
4	Tetrahedral	12
	Dihedral	8
6	Tetrahedral	12
	Octahedral	24
	Dihedral	12
8	Octahedral	24
	Dihedral	16
12	Octahedral	24
	Icosahedral	60
	Dihedral	24
20	Icosahedral	60
	Dihedral	40
30	Icosahedral	60
	Dihedral	60

Theorem B follows at once from Table 2.

Remark. It would be interesting to deduce Theorems A and B without using the canonical regions G^*.

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