POLYNOMIAL DOMINANTS

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Let $f(x)$ be a continuous function of period 2π, and let $F_n(x)$ be a trigonometric polynomial of degree n which dominates $f(x)$ at the points $x_j^n = 2j\pi/(2n+1)$ where $j = 0, \cdots, 2n$, that is, $F_n(x_j^n) \geq |f(x_j^n)|$. The present paper is concerned with relations of inequality between various integral means of the functions $f(x)$ and $F_n(x)$. A similar problem was considered by Hardy and Littlewood [2][1] under the hypothesis that the Fourier series of a function $F(x)$ was a majorant of the Fourier series of $f(x)$. A further variation concerning the means of the Fourier coefficients under the hypothesis of functional dominance, that is, $F(x) \geq |f(x)|$, was treated by the author [1]. In the present work, heavy reliance is made on the results of Marcinkiewicz and Zygmund in their paper *Mean values of trigonometric polynomials* [4]. This paper will be referred to in the sequel as MV.

The results which are obtained are of the type

(1) $$M_a(f) \leq A M_b(F_n) + B\omega(2\pi/n),$$

where (i) $M_a(g)$ represents some integral mean of $g(x)$; (ii) A and B are constants independent of n and the functions $f(x)$ and $F_n(x)$, but dependent on the particular mean M_a; and (iii) $\omega(\delta)$ is the modulus of continuity of $f(x)$.

It can easily be seen that neither term on the right of inequality (1) can be omitted. Suppose first that, for some n, $f(x_j^n) = 0$ but $f(x)$ is not identically zero, and also that $F_n(x_j^n) = 0$ so that $F_n(x) = 0$. Then the left side of (1) will not be zero while the first term on the right vanishes, and consequently the second term on the right is essential. Finally, if the first term were not present in the right member of (1), the left side would be different from zero for non-null functions while the right side would tend to zero as $n \to \infty$.

Throughout the remainder of the paper we shall suppose that $x_j^n = 2j\pi/(2n+1)$; that $\phi_{2n+1}(t) = x_j$ when $x_j \leq t < x_{j+1}$; and that a letter $A, B, \cdots$ with subscript will denote a constant depending only on those parameters appearing in the subscript. The same letter may have a different value at each appearance.

THEOREM 1. *For* $1 < r < \infty$,

[1] Numbers in brackets refer to the references cited at the end of the paper.

(2) $\quad \left(\int_0^{2\pi} |f(x)|^r dx \right)^{1/r} \leqq A_r \left(\int_0^{2\pi} |F_n(x)|^r dx \right)^{1/r} + B_r \omega(2\pi/n),$

where A_r and B_r depend on r alone.

There is a trigonometric polynomial, $T_n(x)$, of degree n such that $|f(x) - T_n(x)| \leqq A\omega(2\pi/n)$, where A is an absolute constant [3, p. 7]. Therefore,

$$\left(\int_0^{2\pi} |f(x)|^r dx \right)^{1/r} \leqq \left(\int_0^{2\pi} |f(x) - T_n(x)|^r dx \right)^{1/r}$$

(3)
$$+ \left(\int_0^{2\pi} |T_n(x)|^r dx \right)^{1/r}$$

$$\leqq A_r \omega(2\pi/n) + \left(\int_0^{2\pi} |T_n(x)|^r dx \right)^{1/r}.$$

Theorem 1 of MV states that

(4) $\quad \left(\int_0^{2\pi} |T_n(x)|^r dx \right)^{1/r} \leqq A_r \left(\int_0^{2\pi} |T_n(x)|^r d\phi_{2n+1}(x) \right)^{1/r}.$

In turn,

$$\left(\int_0^{2\pi} |T_n(x)|^r d\phi_{2n+1}(x) \right)^{1/r}$$

$$\leqq \left(\int_0^{2\pi} |T_n(x) - f(x)|^r d\phi_{2n+1}(x) \right)^{1/r}$$

(5)
$$+ \left(\int_0^{2\pi} |f(x)|^r d\phi_{2n+1}(x) \right)^{1/r}$$

$$\leqq A_r \omega(2\pi/n) + \left(\int_0^{2\pi} |f(x)|^r d\phi_{2n+1}(x) \right)^{1/r}.$$

The hypothesis of dominance at the points $x_j{}^n$, together with Theorem 2 of MV, implies that

$$\left(\int_0^{2\pi} |f(x)|^r d\phi_{2n+1}(x) \right)^{1/r}$$

(6)
$$\leqq \left(\int_0^{2\pi} |F_n(x)|^r d\phi_{2n+1}(x) \right)^{1/r} \leqq A_r \left(\int_0^{2\pi} |F_n(x)|^r dx \right)^{1/r}.$$

The combination of (3) through (6) completes the demonstration of (2).

If $f(x)$ is a trigonometric polynomial of degree n, the approximation process is unnecessary and the terms in $\omega(2\pi/n)$ do not occur.

COROLLARY 1. *If $f(x)$ is a trigonometric polynomial of degree n, then for $1 < r < \infty$*

$$\left(\int_0^{2\pi} |f(x)|^r dx \right)^{1/r} \leq A_r \left(\int_0^{2\pi} |F_n(x)|^r dx \right)^{1/r},$$

where A_r depends on r alone.

THEOREM 2. *For $0 < \mu < 1$,*

$$(7) \qquad \left(\int_0^{2\pi} |f(x)|^\mu dx \right)^{1/\mu} \leq A_\mu \int_0^{2\pi} |F_n(x)| dx + B_\mu \omega(2\pi/n),$$

where A_μ and B_μ depend on μ alone.

Let $T_n(x)$ have the same connotation as above. Then by Theorem 6 of MV,

$$(8) \quad \left(\int_0^{2\pi} |f(x)|^\mu dx \leq \int_0^{2\pi} |f(x) - T_n(x)|^\mu dx + \int_0^{2\pi} |T_n(x)|^\mu dx \right.$$

$$\leq A_\mu [\omega(2\pi/n)]^\mu + B_\mu \left(\int_0^{2\pi} |T_n(x)| d\phi_{2n+1}(x) \right)^\mu.$$

Theorem 1 of MV and the hypothesis of dominance show that

$$\left(\int_0^{2\pi} |T_n(x)| d\phi_{2n+1}(x) \right)^\mu$$

$$\leq \left(\int_0^{2\pi} |T_n(x) - f(x)| d\phi_{2n+1}(x) \right)^\mu$$

$$+ \left(\int_0^{2\pi} |f(x)| d\phi_{2n+1}(x) \right)^\mu$$

$$(9)$$

$$\leq A_\mu [\omega(2\pi/n)]^\mu + \left(\int_0^{2\pi} |F_n(x)| d\phi_{2n+1}(x) \right)^\mu$$

$$\leq A_\mu [\omega(2\pi/n)]^\mu + B_\mu \left(\int_0^{2\pi} |F_n(x)| dx \right)^\mu.$$

Inequalities (8) and (9) therefore imply that

$$\int_0^{2\pi} |f(x)|^\mu dx \leq A_\mu \left(\int_0^{2\pi} |F_n(x)| dx \right)^\mu + B_\mu [\omega(2\pi/n)]^\mu.$$

The extraction of the appropriate root and the corresponding re-labelling of constants then gives (7).

THEOREM 3. *If* $|f(x)| \leq M$, *then*

$$\int_0^{2\pi} |f(x)| \, dx \leq A \log^+ M \int_0^{2\pi} |F_n(x)| \, dx + B$$

$$+ C\omega(2\pi/n)[1 + M + \log^+ M + C\omega(2\pi/n)],$$

where A, B, *and* C *are absolute constants.*

Again, let $T_n(x)$ have the connotation as above. Then by Theorem 5 of MV,

$$\int_0^{2\pi} |f(x)| \, dx$$

$$\leq \int_0^{2\pi} |f(x) - T_n(x)| \, dx + \int_0^{2\pi} |T_n(x)| \, dx$$

$$\leq A\omega(2\pi/n) + B \int_0^{2\pi} |T_n(x)| \log^+ |T_n(x)| \, d\phi_{2n+1}(x) + C$$

(10)

$$\leq B \int_0^{2\pi} \left| |T_n(x)| \log^+ |T_n(x)| - |f(x)| \log^+ |f(x)| \right| d\phi_{2n+1}(x)$$

$$+ A\omega(2\pi/n) + B \int_0^{2\pi} |f(x)| \log^+ |f(x)| \, d\phi_{2n+1}(x) + C$$

$$= S_1 + A\omega(2\pi/n) + S_2 + C.$$

The application of Theorem 1 of MV shows that

$$s_2 \leq B \log^+ M \int_0^{2\pi} |f(x)| \, d\phi_{2n+1}$$

(11)

$$\leq B \log^+ M \int_0^{2\pi} |F_n(x)| \, d\phi_{2n+1}(x)$$

$$\leq B \log^+ M \int_0^{2\pi} |F_n(x)| \, dx.$$

To obtain an estimate for S_1, first suppose that for a particular value of j either $|T_n(x_j^n)| \leq 1$ or $|f(x_j^n)| \leq 1$. If the former holds, then $|f(x_j^n)| \leq 1 + A\omega(2\pi/n)$, and $|f(x_j^n)| \log^+ |f(x_j^n)| \leq MA\omega(2\pi/n)$. If the latter holds, then $|T_n(x_j^n)| \leq \min[M + A\omega(2\pi/n), 1 + A\omega(2\pi/n)]$, so that $|T_n(x_j^n)| \log^+ |T_n(x_j^n)| \leq [M + A\omega(2\pi/n)]A\omega(2\pi/n)$.

Suppose next that for a particular value of j, both $|T_n(x_j^n)| > 1$ and $|f(x_j^n)| > 1$. For brevity suppose also that $|T_n(x_j^n)| \geq f(x_j^n)$. The other case will lead to the same conclusion. Then

$$
\begin{aligned}
|T_n(x_j^n)| \log^+ |T_n(x_j^n)| &- |f(x_j^n)| \log^+ |f(x_j^n)| \\
&= |T_n(x_j^n)| \, [\log^+ |T_n(x_j^n)| - \log^+ |f(x_j^n)|\,] \\
&\quad + \log^+ |f(x_j^n)| \, [\,|T_n(x_j^n)| - |f(x_j^n)|\,] \\
&\leq |T_n(x_j^n)| \log [1 + |T_n(x_j^n)| - |f(x_j^n)|\,] + A\omega(2\pi/n) \log^+ M \\
&\leq [M + A\omega(2\pi/n)]A\omega(2\pi/n) + A\omega(2\pi/n) \log^+ M.
\end{aligned}
$$

In any event, the individual terms comprising the mean in S_1 are less than $A\omega(2\pi/n)[M + \log^+ M + A\omega(2\pi/n)]$, and therefore S_1 has the same bound. The theorem then follows from (10) and (11).

REFERENCES

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