PROPERTIES EQUIVALENT TO THE COMPLETENESS OF $\{e^{-t}t^{\lambda_n}\}$

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We are concerned with the following three properties which may be possessed by an increasing sequence $\{\lambda_n\}$ of positive integers.

(A) If $\{a_n\}$ is a sequence of complex numbers such that, for some β, $a_n = O(n^\beta)$ and $\Delta^{\lambda_n} a_0 = 0$ $(n=1, 2, \cdots)$, a_n is a polynomial in n; here

$$\Delta^n a_0 = \sum_{k=0}^{n} (-1)^k C_{n,k} a_k.$$

(B) The set $\{t^{\lambda_n} e^{-ct}\}$ is complete $L^2(0, \infty)$; that is,

$$\int_0^\infty t^{\lambda_n} e^{-ct} \phi(t) dt = 0 \qquad (n = 1, 2, \cdots; \phi \in L^2)$$

implies $\phi(t) = 0$ almost everywhere.[1]

(C) If $f(z)$ is regular and $O(|z|^\alpha)$ for some α in the half-plane $x > -\epsilon$, $\epsilon > 0$, and $f^{(\lambda_n)}(0) = 0$ $(n=1, 2, \cdots)$, $f(z)$ is a polynomial.[2]

W. H. J. Fuchs [3][3] showed that (A) and (B) are equivalent. We shall give a somewhat simpler proof, and show in addition that (C) is equivalent to (A) and (B).

Fuchs showed that (A) is true if $n(r) \geqq r/2 - \gamma$ for some constant γ, where $n(r)$ is the number of $\lambda_n \leqq r$. R. P. Agnew discovered independently [1] that (A) is true if $\lambda_n = 2n$; a simplified proof given by Pollard [5] was the starting point of this note. Boas [2] has shown by other methods that it is enough to have $n(r) \geqq r/2 - r\delta(r)$, where $\int^\infty r^{-1}\delta(r)dr$ converges and $\delta(r)$ satisfies some mild auxiliary conditions. (Fuchs, in a paper [3a] which appeared while this note was in the press, has shown that a necessary and sufficient condition for (A) is that $\int^\infty r^{-2}\psi(r)dr$ diverges, where $\log \psi(r) = 2\sum_{\lambda_n \leqq r} \lambda_n^{-1}$.)

Let $P(\lambda_n)$ mean that $\{\lambda_n\}$ has property (P); $P(\lambda_n - N)$, that the sequence $\{\lambda_n - N\}$ has (P), where $\lambda_n - N$ is replaced by 0 if $\lambda_n < N$. Our line of reasoning is schematically as follows: $A(\lambda_n) \to B(\lambda_n) \to C(\lambda_n + N) \to A(\lambda_n + N) \to A(\lambda_n - N) \to B(\lambda_n - N) \to C(\lambda_n) \to A(\lambda_n)$. It would be more direct to use $B(\lambda_n) \to B(\lambda_n - N)$; this can be quoted from the

Received by the editors December 17, 1945.

[1] Replacing ct by t, we see that (B) is independent of c.

[2] (C) thus concerns the analytic continuation of a function defined by a lacunary power series $\sum c_n z^{\mu_n}$, where $\{\mu_n\}$ is the sequence of positive integers complementary to $\{\lambda_n\}$.

[3] Numbers in brackets refer to the references at the end of the paper.

work of Fuchs, but the proof is rather involved, and we know of no really simple direct proof. One implication of our reasoning is that "$B(\lambda_n)\rightarrow B(\lambda_n-N)$" is actually equivalent to our other results, and not merely a convenient lemma. Carrying out the indicated scheme actually involves only four nontrivial steps.

(1) $A(\lambda_n)\rightarrow A(\lambda_n-N)$. It is sufficient to prove this when $N=1$. Suppose that $\{\lambda_n-1\}$ does not have (A); we then have a nonpolynomial sequence $\{a_n\}_{n=0}^{\infty}$, $a_n=O(n^\beta)$, $\Delta^{\lambda_n-1}a_0=0$ if $\lambda_1>0$, $a_0=0$ if $\lambda_1=0$. Consider the sequence $\{b_n\}_{n=0}^{\infty}$, where $b_0=0$, $b_n=-\sum_{k=0}^{n-1}a_k$ for $n=1, 2, \cdots$. Then $a_n=b_n-b_{n+1}$, and for $p>0$, by a simple direct computation, $\Delta^p a_0=\Delta^{p+1}b_0$. Consequently $\Delta^{\lambda_n}b_0=\Delta^{\lambda_n-1}a_0=0$ if $\lambda_n>0$, $\Delta^0 b_0=b_0=0$; furthermore, if $\{b_n\}$ were a polynomial sequence, $\{a_n\}$ would be one also; and $b_n=O(n^{\beta+1})$. Hence $\{\lambda_n\}$ cannot have (A) if $\{\lambda_n-1\}$ does not.

(2) $A\rightarrow B$. Suppose that $\phi(t)\in L^2$ and

$$\int_0^\infty e^{-t/2}t^{\lambda_n}\phi(t)dt = 0, \qquad n = 1, 2, \cdots .$$

We have to show that $\phi(t)=0$ almost everywhere if (A) is true. We define b_n by

$$n!b_n = \int_0^\infty e^{-t/2}t^n\phi(t)dt;$$

then

$$\text{(I)}\qquad \begin{aligned} a_n = \Delta^n b_0 &= \int_0^\infty e^{-t/2}\phi(t)\left\{\sum_{k=0}^{n}(-1)^k C_{n,k}t^k/k!\right\}dt \\ &= \int_0^\infty e^{-t/2}L_n(t)\phi(t)dt, \end{aligned}$$

where $L_n(t)$ is the nth Laguerre polynomial. Thus

$$|a_n|^2 \leqq \left\{\int_0^\infty e^{-t}L_n^2(t)dt\right\}\left\{\int_0^\infty |\phi(t)|^2dt\right\} = \int_0^\infty |\phi(t)|^2dt,$$

and so $a_n=O(1)$. Since, as is readily verified, $b_n=\Delta^n a_0$, (A) implies that $\{a_n\}$ is a polynomial sequence, which must be constant since $\{a_n\}$ is bounded. Hence $a_n=a_0$ for $n=1, 2, \cdots$. Since $e^{-t/2}L_n(t)$ is orthonormal, $\sum a_n^2$ converges, by (I). But this is possible only if all the a_n vanish. Hence $b_n=0$, $n=0, 1, \cdots$. But then $\phi(t)=0$ almost everywhere, since $B(n-1)$ is true.[4]

(3) $B(\lambda_n)\rightarrow C(\lambda_n+N)$, $N\geqq\alpha+1$. Suppose that $f(z)$ satisfies the hy-

[4] This is equivalent to the completeness of the set $\{e^{-t}L_n(t)\}$; see [6, p. 104].

potheses of (C), with λ_n+N; we have to show that $f(z)$ is a polynomial if (B) is true for $\{\lambda_n\}$. For convenience, suppose $\epsilon=2$. If $P(z)$ is the sum of the terms through z^N in the Maclaurin series of $f(z)$, $z^{-N-1}\{f(z)-P(z)\}$ belongs to the class $H^2(-1)$ of functions $g(z)$ such that $g(x+iy)\in L^2$, qua function of y, uniformly in $x\geqq -1$, and consequently [4, p. 8]

$$f(z) = P(z) + z^N \int_0^\infty e^{-zt}e^{-t}\phi(t)dt, \qquad \phi \in L^2,\ x > -1.$$

Since $f^{(\lambda_n+N)}(0)=0$,

$$\int_0^\infty t^{\lambda_n}e^{-t}\phi(t)dt = 0, \qquad \lambda_n \geqq N.$$

Since (B) is assumed for $\{\lambda_n\}$, $\phi(t)=0$ almost everywhere and so $f(z)\equiv P(z)$.

(4) $C\rightarrow A$. Let $a_n=O(n^\beta)$, $\Delta^{\lambda_n}a_0=0$; we may assume that β is an integer. Define $b_n=\Delta^n a_0$, so that $b_n=O(n^\beta 2^n)$, $b_{\lambda_n}=0$. Consider

$$\begin{aligned} f(z) = \sum_{n=0}^\infty b_n z^n &= \sum_{n=0}^\infty z^n \sum_{k=0}^\infty (-1)^k C_{n,k} a_k \\ &= \sum_{k=0}^\infty (-1)^k a_k \sum_{n=k}^\infty C_{n,k} z^n \\ &= \frac{1}{1-z}\sum_{k=0}^\infty a_k\left(\frac{z}{1-z}\right)^k. \end{aligned}$$

The first series for $f(z)$ converges for $|z|<1/2$; the last, for $|z/(1-z)|<1$, that is, for $x<1/2$. Consequently $f(z)$ is regular in this half-plane. There is a number K such that $|a_n|\leqq Kn(n-1)\cdots(n-\beta+1)$, $n\geqq\beta$. We then have

$$\begin{aligned} |f(z)| &\leqq \left|\frac{1}{1-z}\sum_{k=0}^{\beta-1} a_k\left(\frac{z}{1-z}\right)^k\right| + K\sum_{k=\beta}^\infty \frac{k!}{(k-\beta)!}\left|\frac{z}{1-z}\right|^k \\ &\leqq O(1) + \frac{K\beta!}{|1-z|}\left|\frac{z}{1-z}\right|^\beta\left(1-\left|\frac{z}{1-z}\right|\right)^{-\beta-1} = O(|z|^\beta) \end{aligned}$$

in $x<1/2-\epsilon$, $\epsilon>0$. Since (C) is assumed, $f(-z)$ is a polynomial. Hence all b_n vanish from some n_0 on, and

$$a_n = \Delta^n b_0 = \sum_{k=0}^{n_0}(-1)^k b_k n(n-1)\cdots(n-k+1)/k!,$$

a polynomial in n.

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