A NOTE ON THE RIEMANN ZETA-FUNCTION

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Let $\rho_\nu = \beta_\nu + i\gamma_\nu$ be the zeros of the Riemann zeta-function $\zeta(1/2+z)$ whose real part $\beta_\nu \geqq 0$. Then we have the following formula which is an improvement on Paley-Wiener's [**1**, p. 78][1]

$$\int_1^T \frac{\log |\zeta(1/2+it)|}{t^2}\,dt = 2\pi\sum_{\nu=1}^{\infty} \frac{\beta_\nu}{|\rho_\nu|^2} + \int_0^{\pi/2} R\{e^{-i\theta}\log\zeta(1/2+e^{i\theta})\}\,d\theta + O\left(\frac{\log T}{T}\right).$$

In order to prove this formula let ρ_ν $(\nu=1, 2, \cdots, n)$ be the n zeros of $\zeta(1/2+z)$ for which $0<\gamma_\nu<T$ and $0\leqq\beta_\nu<1/2$. We require the following lemma:

LEMMA. *Let K be the unit semicircle with center $z=0$ lying in the right half-plane $R(z)>0$ and let C be the broken line consisting of three segments L_1 $(0\leqq x\leqq T,\ y=T)$, L_2 $(0\leqq x\leqq T,\ y=-T)$ and L_3 $(x=T,\ -T\leqq y\leqq T)$. Then*

$$\frac{1}{\pi}\int_1^T \frac{\log |\zeta(1/2+it)|}{t^2}\,dt = 2\sum_{\nu=1}^{n} \frac{\beta_\nu}{|\rho_\nu|^2} + \frac{1}{2\pi i}\int_K \frac{\log\zeta(1/2+z)}{z^2}\,dz - \frac{1}{2\pi i}\int_C \frac{\log\zeta(1/2+z)}{z^2}\,dz. \tag{1}$$

This is a form of Carleman's theorem which can be proved by a method of proof analogous to that of Littlewood's theorem (Titchmarsh [**3**, pp. 130–134]).

Let Γ be a contour describing C, K and the corresponding part of the imaginary axis, and let ρ_ν be a point interior to Γ, and $\log(z-\rho_\nu)$ be taken as its principal value. We write C_1 as a contour describing Γ in positive direction to the point $i\gamma_\nu$, then along the segment $y=\gamma_\nu$, $0<x<\beta_\nu-r$, and describing a small circle with center $z=\rho_\nu$, radius r, then going back along the negative side of this segment to $i\gamma_\nu$, and then along Γ to the starting point.

By Cauchy's theorem we get

$$\int_{C_1} \frac{\log(z-\rho_\nu)}{z^2}\,dz = 0.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Hence

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\log(z-\rho_\nu)}{z^2}\,dz=-\int_0^{\beta_\nu}\frac{dx}{(x+i\gamma_\nu)^2}$$

where the integral round the small circle with center $z=\rho_\nu$, radius r, tends to zero as $r\to 0$. This formula is also true for $\beta_\nu=0$.

Put $\zeta(1/2+z)=\phi(z)\prod_{\nu=1}^{n}(z-\rho_\nu)\prod_{\nu=1}^{n}(z-\bar\rho_\nu)$ where $\phi(z)$ is regular and has no zero in and on Γ. Then we get

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\log\zeta(1/2+z)}{z^2}\,dz=\sum_{\nu=1}^{n}\left(\frac{1}{\rho_\nu}-\frac{1}{i\gamma_\nu}\right)+\sum_{\nu=1}^{n}\left(\frac{1}{\bar\rho_\nu}+\frac{1}{i\gamma_\nu}\right)$$
$$=2\sum_{\nu=1}^{n}\frac{\beta_\nu}{|\rho_\nu|^2}.$$

From this the lemma follows.

Now we have

$$(2)\qquad \int_C\frac{\log\zeta(1/2+z)}{z^2}\,dz=-\int_{L_1}+\int_{L_2}+\int_{L_3}.$$

On account of

$$\log\zeta(1/2+x+iT)=O(1)\qquad\text{for } x\geqq 1$$

we have

$$(3)\qquad \int_{L_1}=\int_0^1\frac{\log\zeta(1/2+x+iT)}{(x+iT)^2}\,dx+O\left(\frac{1}{T}\right).$$

Since (Titchmarsh [2, p. 5])

$$\arg\zeta(1/2+x+iT)=O(\log T)\qquad\text{for } 0\leqq x\leqq 1$$

and (Titchmarsh [2, p. 59])

$$\log|\zeta(1/2+x+iT)|$$
$$=\frac{1}{2}\sum_{|\gamma-T|<1}\log\{(x-\beta)^2+(T-\gamma)^2\}+O(\log T),$$

then

$$(4)\qquad \int_0^1\frac{\log\zeta(1/2+x+iT)}{(x+iT)^2}\,dx=O\left(\frac{\log T}{T^2}\right).$$

From (3) and (4) we get

$$(5)\qquad \int_{L_1}=O\left(\frac{\log T}{T}\right).$$

Similarly

$$\int_{L_2} = O\left(\frac{\log T}{T}\right). \tag{6}$$

Since $\log \zeta(1/2+T+iy) = O(2^{-T})$, we get

$$\int_{L_3} = O(T2^{-T}). \tag{7}$$

By (1), (2), (5), (6) and (7) we have

$$\begin{aligned}\int_1^T \frac{\log |\zeta(1/2+it)|}{t^2}\,dt &= 2\pi \sum_{\nu=1}^{n} \frac{\beta_\nu}{|\rho_\nu|^2} \\ &\quad + \frac{1}{2i}\int_K \frac{\log \zeta(1/2+z)}{z^2}\,dz + O\left(\frac{\log T}{T}\right).\end{aligned} \tag{8}$$

But (Ingham [4, p. 70])

$$\sum_{\nu=n+1}^{\infty} \frac{\beta_\nu}{|\rho_\nu|^2} = O\left(\sum_{\gamma>T} \frac{1}{\gamma^2}\right) = O\left(\frac{\log T}{T}\right). \tag{9}$$

The formula follows from (8) and (9).

Finally, if we make $T \to \infty$ then

$$\int_1^\infty \frac{\log |\zeta(1/2+it)|}{t^2}\,dt = \int_0^{\pi/2} R\{e^{-i\theta}\log \zeta(1/2+e^{i\theta})\}\,d\theta$$

gives a necessary and sufficient condition for the truth of the Riemann hypothesis.

References

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2. E. C. Titchmarsh, *The zeta-function of Riemann*, Cambridge, 1930.

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