MOEBIUS TRANSFORMATIONS AND CONTINUED FRACTIONS

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R. E. Lane[1] has shown how the criterion for convergence of a periodic continued fraction can be based upon simple properties of the linear fractional transformation (Moebius transformation)

$$Z = A(z) = (az + b)/(cz + d). \tag{1}$$

The method may be further adapted to the pure "transformation point of view" by: (a) interpretation of the two lemmas of Lane's paper as reductions of (1) to its similarity normal forms; (b) proving and expressing the final theorem in terms of the theory of the Moebius transformation only.

This will be done in the present note which can be read without reference to Lane's paper.

The Moebius transformation[2] (1) is defined by its matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of complex elements with nonvanishing determinant δ, or by any matrix λA where $\lambda \neq 0$. Let x_1, x_2 be its fixed points: $A(x_\nu) = x_\nu$; then these are also the fixed points of the iterated transformations $z_n = A^n(z)$ $(n = 1, 2, \cdots)$.

Let T be the matrix of another Moebius transformation $z' = T(z)$ by which the new variable z' is introduced. Then $T(x_\nu)$ are the fixed points of the transformation[3]

$$Z' = TAT^{-1}(z'). \tag{2}$$

Suppose $x_1 \neq x_2$; then $T(x_1) \neq T(x_2)$ and the transformation T may be chosen so that $T(x_1) = 0$, $T(x_2) = \infty$, namely, $z' = (z - x_1)/(z - x_2)$, and (2) becomes

$$Z' = kz' \tag{3}$$

where $k = (a - cx_1)/(a - cx_2)$ is a complex constant, uniquely defined

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[1] R. E. Lane, *The convergence and the values of periodic continued fractions*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 246–250.

[2] Cf. C. Carathéodory, *Conformal representation*, Cambridge, 1932, chap. 1, §26.

[3] The matrix of the product (that is, composition) of two Moebius transformations, first A, then B, is the product BA of the two matrices.

by A and the same for all similar transformations SAS^{-1}. The transformation A is said to be *elliptic* if $|k| = 1$ $(k \neq 1)$, *hyperbolic* if $k > 0$, in all other cases *loxodromic*; it is the identity if and only if $k = 1$.

If $x_1 = x_2 = x$, a transformation T can be found so that $T(x) = \infty$; then TAT^{-1} becomes a translation $Z' = z' + b'$. By suitable choice of T one can reach the normal form

$$Z' = z' + 1 \tag{4}$$

of A which, in this case, is said to be *parabolic*.

For a given $z \neq x_\nu$ $(\nu = 1, 2)$ we consider the recurring sequence $z_n = A(z_{n-1}) = \cdots = A^n(z)$. If it is convergent its limit is a fixed point x_1 of A. If $z_n \to x_1$ for all z of a neighborhood of x_1, we call x_1 an *attractive* fixed point of A. At the same time $T(x_1)$ is an attractive fixed point of TAT^{-1}. If $x_1 \neq x_2$ it follows from (3) that x_1 is attractive if and only if $|k| < 1$, and then $z_n \to x_1$ for all $z \neq x_2$; therefore the fixed point x_2 is said to be *repulsive*.

Evidently ∞ is an attractive fixed point of the translation (4); thus the only fixed point x of a parabolic transformation A is always attractive.[4]

If A is elliptic, $|k| = 1$, the sequence z_n is bounded and divergent whatever the initial point $z \neq x_\nu$; the points z_n all lie on the circumference of the same invariant circle of A. The fixed points x_1, x_2 are symmetric with respect to this circle; they are called *indifferent* fixed points of A.

Now let

$$f = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots$$

be a given continued fraction where all $a_\mu \neq 0$. Its approximants f_q may be represented in the following way: Let

$$A_1 = \begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & a_2 \\ 1 & b_2 \end{pmatrix}, \cdots .$$

Then one has

$$f_0 = A_1(\infty) = 0, \qquad f_1 = a_1/b_1 = A_1(0) = A_1A_2(\infty), \cdots ,$$
$$f_{q-1} = A_1A_2 \cdots A_{q-1}(0) = A_1A_2 \cdots A_q(\infty).$$

By definition let

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[4] In the book of G. Julia, *Principes géométriques d'analyse*, I, Paris, 1930, it is stated on p. 30 that the fixed point of a parabolic transformation is indifferent. This depends, however, on the inadequate definition of "indifferent" given there on p. 23.

$$f_{-1} = \infty .$$

Now let f be periodic with a period of length m so that for all $n=0, 1, 2, \cdots$

$$a_q = a_r, \qquad b_q = b_r \quad \text{if} \quad q = nm + r$$

where r may be the least non-negative remainder (mod m) of q. We introduce the "period matrix"

$$A = \lambda A_1 A_2 \cdots A_m \qquad (\lambda \neq 0).$$

Then

$$f_{q-1} = A^n A_1 \cdots A_r(\infty) = A^n(f_{r-1})$$

for $r=0, 1, \cdots, m-1$, whence the main result follows immediately:

The periodic continued fraction f is convergent if and only if the following three conditions are satisfied:

1. *The Moebius transformation with the period matrix A is not elliptic.*
2. *The attractive fixed point x_1 of A is finite.*
3. *For all $r=0, \cdots, m-1$ one has $f_{r-1} \neq x_2$ where x_2 is the repulsive fixed point of A.*

Then $f=x_1$.

Remark. For $r=0$ the condition 3 implies that also x_2 must be finite. There is no restriction for f_{r-1} if A is parabolic. The condition 1 implies Lane's inequality (1, 4). In fact $f_{m-1}=A(\infty)$ is the pole of the transformation A^{-1} which has equal distance from both fixed points x_1, x_2 if and only if A is elliptic. In fact from (3) one has $f_{m-1}-x_1 = k(f_{m-1}-x_2)$ whence moreover it follows at once that in the non-elliptic case the pole of A^{-1} is nearer to the attractive fixed point of A.

For the application of the theorem it may be useful to have the necessary and sufficient condition for a Moebius transformation

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to be elliptic; it is

$$\Re(\bar{\tau} \cdot (\tau^2 - 4\delta)^{1/2}) = 0$$

where $\tau = a+d$, $\delta = ad-bc$, and τ the conjugate complex of τ.

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