CONCERNING THE SEPARABILITY OF CERTAIN LOCALLY CONNECTED METRIC SPACES

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If a connected metric space S is locally separable, then S is separable.[1] If a connected, *locally connected*, metric space S is locally *peripherally* separable, then S is separable.[2] Furthermore if a connected, locally connected, *complete* metric space S satisfies certain "flatness" conditions, it is known to be separable.[3] These "flatness" conditions are rather strong and involve both im kleinen and im grossen properties, which makes application rather awkward in some cases. If, however, this space S contains no skew curve[4] of type 1, then S has a certain amount of "flatness," but not quite enough to imply separability as can be seen from the following example. Let S consist of the points of the 2-sphere, distance being redefined as follows: (1) if the points X and Y of S lie on the same great circle through the poles, then $d(X, Y)$ is the ordinary distance on the sphere but (2) if the points lie on different great circles through the poles, then $d(X, Y)$ is the sum of the ordinary distances from each point to the same pole, using the pole which gives the smaller sum. The space S is a connected, locally connected, complete metric space which contains no skew curve of type 1 but S is not separable. Furthermore, S contains no cut point. However, if this last condition is strengthened slightly, separability follows as is seen in the following theorem.

THEOREM 1. *Let S denote a locally connected, complete metric space such that no pair of points cuts S. If S contains no skew curve of type* 1, *then S is separable.*

PROOF. Suppose, on the contrary, that S is not separable. Let T_0

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[1] Paul Alexandroff, *Über die Metrization der im kleinen kompakten topologischen Räume*, Math. Ann. vol. 92 (1924) pp. 294–301. Also W. Sierpinski, *Sur les espaces métriques localement separables*, Fund. Math. vol. 21 (1933) pp. 107–113.

[2] F. B. Jones, *A theorem concerning locally peripherally separable spaces*, Bull. Amer. Math. Soc. vol. 41 (1935) pp. 437–439.

[3] F. B. Jones, *Concerning certain topologically flat spaces*, Trans. Amer. Math Soc. vol. 42 (1937) pp. 53–93, Theorem 31. Also F. B. Jones, Bull. Amer. Math. Soc. Abstract 47-1-93.

[4] Kuratowski in his paper, *Sur le problème des courbes gauches en Topologie*, Fund. Math. vol. 15 (1930) pp. 271–283, defined two "skew curves." One of type 1 is topologically equivalent to the sum of three simple triods each two of which intersect precisely at their end points.

denote a simple triod in S and let $M_1=T_0$. Let T_1 denote a simple triod in S having only its end points in M_1, and let $M_2=T_0+T_1$. Let T_2 denote a simple triod in S having only its end points in M_2, and let $M_3=T_0+T_1+T_2$. This process may be continued, so that if z is an ordinal less than Ω_1, then (a) $M_z=\overline{\Sigma T_\eta}$, $0\leqq\eta<z$, (b) M_z is a separable (hence proper) subcontinuum of S, and (c) T_z is a simple triod having only its end points in M_z. Hence $T_1, T_2, T_3, \cdots, T_z, \cdots$ is an uncountable sequence α of simple triods such that no two of them have a point in common which is not an end point of one of them. For each $z<\Omega_1$, let d_z denote the smallest of the distances: $d(A, \text{arc } BOC)$, $d(B, \text{arc } AOC)$, and $d(C, \text{arc } AOB)$, where A, B, and C are the end points and O is the emanation point of T_z. Let H_1 denote the set of all simple triods T such that for some z, T is T_z of α and each end point of T lies together with a point of M_1 in a connected domain of diameter less than $d_z/5$.

Suppose that H_1 is uncountable. There exists a positive number ϵ such that for uncountably many different ordinals $z<\Omega_1$, T_z belongs to H_1 and $1.1\epsilon>d_z/5>\epsilon$. But since M_1 is separable, there exist three distinct points X_1, X_2, and X_3 and three ordinals $\alpha<\beta<\gamma<\Omega_1$ such that (1) T_α, T_β, and T_γ each belong to H_1, (2) for each ξ, $\xi=\alpha, \beta, \gamma$, $1.1\epsilon>d_\xi/5>\epsilon$, and (3) for each ξ, $\xi=\alpha, \beta, \gamma$, and each i, $i=1, 2, 3$, there exists a connected domain $D_{\xi i}$ which contains X_i and an end point of T_ξ, and whose diameter is less than $d_\xi/5$. Now for each ξ, $\xi=\alpha, \beta, \gamma$, let O_ξ denote the emanation point of T_ξ. From (2), (3), the definition of d_z, and the triangle axiom on the distance function, it follows that the connected domains, $D_1=\Sigma D_{\xi 1}$, $D_2=\Sigma D_{\xi 2}$, and $D_3=\Sigma D_{\xi 3}$, are mutually exclusive and neither $\overline{D}_1$, $\overline{D}_2$, nor $\overline{D}_3$ contains either O_α, O_β, or O_γ. Because of the restricted way in which the triods may intersect, no point outside of $D_1+D_2+D_3$ lies in more than one of the triods T_α, T_β, and T_γ. But $T_\alpha\cdot D_1$ may be joined to $T_\beta\cdot D_1$ by an arc in D_1; $T_\alpha\cdot D_1$ may be joined to $T_\gamma\cdot D_1$ by an arc in D_1; $T_\alpha\cdot D_2$ may be joined to $T_\beta\cdot D_2$ by an arc in D_2; and so on; and in the sum of these arcs together with $T_\alpha+T_\beta+T_\gamma$ there exists a skew curve of type 1. So the assumption that H_1 is uncountable leads to a contradiction. Hence H_1 is countable.

Let z_2 denote the smallest ordinal such that if $z\geqq z_2$, T_z of α does not belong to H_1. Evidently $z_2<\Omega_1$. Let H_2 denote the set of all triods T such that for some z, T is T_z of α and each end point of T lies together with a point of M_{z_2} in a connected domain of diameter less than $d_z/5$. The collection H_2 is countable. Let z_3 denote the smallest ordinal such that if $z\geqq z_3$, T_z of α does not belong to H_2. Evidently $z_3<\Omega_1$. Let H_3 denote the set of all triods T such that for some z, T is T_z of α and each end point of T lies together with a point of M_{z_3} in a connected domain of diameter less than $d_z/5$. The collection H_3 is countable. Continue this process, so that for each natural number n, H_n is defined and countable. There exists an ordinal number $z<\Omega_1$ such that for each n, $z>z_n$. Let $\bar{z}$ denote the first such ordinal. Clearly, $T_{\bar{z}}$ of α does not belong to H_n for any n. Let D_1, D_2, and D_3 denote three mutually exclusive connected domains covering respectively the end points of $T_{\bar{z}}$ such that each has a diameter less than $d_{\bar{z}}$. Since $M_1\subset M_2\subset M_3\subset \cdots \subset M_z\subset \cdots$ and $T_{\bar{z}}$ has its end points in $M_{\bar{z}}$, there exists an integer i such that each of the domains, D_1, D_2, and D_3, intersects M_{z_i}. Hence $T_{\bar{z}}$ belongs to H_i. This is a contradiction.

THEOREM 2. *Let S denote a locally connected, complete metric space such that no pair of points cuts S. If S does not contain uncountably many skew curves of type* 1, *then S is separable.*

Theorem 2 may be established by the argument for Theorem 1, taking for T_0 the closure of the set consisting of all points X such that X belongs to a skew curve of type 1 lying in S. The connectedness of M_z was not used in the argument.

Comment. This result (Theorem 1) cannot be extended to complete Moore spaces.[5] For a locally connected complete Moore space exists which is not cut by any pair of its points and which contains no skew curve of type 1 but which nevertheless is not separable.[6] Furthermore, a separable such space exists which is not completely (perfectly) separable and hence is not metric.[7] The relation between Moore and metric spaces (in this connection) is shown in Theorem 3.

THEOREM 3. *Let M denote a locally connected, complete Moore space such that* (1) *no pair of points cuts M and* (2) *M contains no skew curve of type* 1. *In order that M be metric it is necessary and sufficient that M be completely* (*perfectly*) *separable.*

PROOF. Since any metric, complete Moore space is a complete metric space[8] and any separable metric space is completely separable, the necessity of the condition follows at once from Theorem 1. Since a

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[5] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloqiuum Publications, vol 13, 1932. Hereinafter this book will be referred to as *Foundations.* A complete Moore space is a space satisfying Axioms 0 and 1 of *Foundations.*

[6] R. L. Moore, *Concerning separability*, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 56–58, Example 1.

[7] Ibid. Example 2.

[8] J. H. Roberts, *A property related to completeness*, Bull. Amer. Math. Soc. vol. 38 (1932) pp. 835–838.

Moore space is a regular Hausdorff space, the sufficiency of the condition is well known.[9]

THEOREM 4. *Every metric space satisfying Axioms* 0–4 *of R. L. Moore's Foundations is completely* (*perpectly*) *separable.*

PROOF. Let S be a metric space satisfying Axioms 0–4 of *Foundations*. No finite set of points separates S.[10] Furthermore, with the help of Theorem 7 of Chapter III of *Foundations* it can be shown that S contains no skew curve of type 1. It follows from the preceding theorem that S is completely separable.

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[9] P. Urysohn, *Zum Metrisationsproblem*, Math. Ann. vol. 94 (1925) pp. 309–315, and A. Tychonoff, *Über einen Metrisationssatz von P. Urysohn*, Math. Am. vol. 95 (1926) pp. 139–142. See *Foundations*, p. 464.

[10] F. B. Jones, *Certain consequences of the Jordan curve theorem*, Amer. J. Math. vol. 63 (1941) pp. 531–544, Theorem 25.