or to numerical coefficients for arbitrary degree differences. (Received January 16, 1946.)

61. Leonard Tornheim: *A method for determining the number of compositions of certain types.*

In the remainder on division of x^m by $x^n - a_1x^{n-1} - \cdots - a_n$, the coefficient of x^{n-1} is $\sum a_{i_1}a_{i_2}\cdots$, where the sum is over all compositions $(i_1, i_2, \cdots)$ of $m-n+1$ with every element $i \leqq n$. By assigning to $a_1, \cdots, a_n$ certain numerical values the total number of compositions with elements $i \leqq n$ of the following types are obtained: (1) all such compositions; (2) all with certain elements absent; (3) all containing certain elements a given number of times; and (4) all with (2) and (3) holding jointly. Also it is possible with this method to find the total number of partitions with parts not greater than n, to list them, and to determine how many compositions correspond to a given partition. (Received January 17, 1946.)

Analysis

62. Gertrude Blanch: *On the computation of Mathieu functions.*

If a characteristic value of Mathieu's differential equation is known to within an error λ, it is possible, by fairly simple means, to correct the characteristic value and the approximate values of the Fourier coefficients defining the solution, to within an error proportional to λ^2. The precise formulas for the corrections, a systematic method of carrying out the computations, and two illustrative examples are given in this paper. Although Mathieu functions only are dealt with, the method is applicable to solutions of other types, where the coefficients of the Fourier (or power) series are determined by a three-term recurrence formula. (Received January 12, 1946.)

63. R. C. Buck: *On a class of entire functions.* I.

Let $K(a, c)$ denote the class of entire functions of order 1 and of type at most a on the whole real axis and type c on the whole imaginary axis, with $c < \pi$. This class has the property that a function $f(z)$ belonging to it is determined completely by the sequence $f(n)$, $n = 1, 2, \cdots$. The author first proves a necessary and sufficient condition that for a given sequence w_n there exist a function $f(z)$ of $K(a, c)$ such that $f(n) = w_n$. Using this, a wide variety of theorems is obtained. For example, if $f(n) = r_n e^{i\theta_n}$ and if there is an α, $c < \alpha \leqq \pi$, such that $\lim\inf\{\cos(\theta_n + n\alpha)\}^{1/n} > 0$, then $f(z) \equiv 0$, if $f(z)$ belongs to $K(a, c)$. As a special case, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two disjoint closed convex sets and if $f(n) \in \mathcal{D}_1$ for $n \in \mathcal{A}_1$, $f(n) \in \mathcal{D}_2$ for $n \in \mathcal{A}_2$ where $\mathcal{A}_1 \cup \mathcal{A}_2$ has density 1, and if $\mathcal{A}_1$ has density 1/2 and satisfies an additional condition, then $\lim\sup \log|f(iy)|/|y| \geqq \pi$. In case the w_n are close enough to integral values, more can be said. Thus, for example, it is proved that no function of $K(0, 0) = K_0$ can take prime values at the integers without being constant. (Received January 17, 1946.)

64. R. C. Buck: *On a class of entire functions.* II.

The results of the previous paper are extended to more general sequences than $\{f(n)\}$, treating instead numbers F_n expressible as linear combinations of the numbers $f(n)$. Most of the theorems proved for the sequence $f(n)$ hold also for F_n. Thus, for example, if $F_n = \sum_0^n g(k)f(k)$ where $g(z)$ belongs to K_0 and $f(z)$ to $K(a, c)$ for $c < \pi$, then for any $\epsilon > 0$, $\log|F_n| > (h(0) - \epsilon)n$ for a set of maximum density at least $1 - c/\pi$. These theorems also have corollaries concerned with the solution of certain functional equations. Extension is also made to sequences based on $f(\lambda_n)$ where $\limsup|n-\lambda_n|^{1/n}<1$. Counterexamples show that for some of these theorems, the condition $\lambda_n-n=O(1)$ is not sufficient. (Received January 17, 1946.)

65. Herman Chernoff: *Complex solutions of partial differential equations. II.*

The author considers certain "classes" $\mathcal{C}$ of complex solutions of equations $\Delta u+Au_x+Bu_y+Cu=0$. (For a more detailed discussion of "classes" see Bergman, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155 and vol. 57 (1945) pp. 299–331.) He defines in the usual way the index of points where $u(x, y)=\alpha$. A function u which has the property that the sum of the indices in the circle $|z|<r$ is 0 or 1 for each value of α and for all $r<\rho$ is said to be *pseudo-simple in the circle* $|z|<\rho$. Generalizing the classical results on simple functions of a complex variable, the author proves the following inequalities: if $u(x, y)=\sum A_{mn}z^m\bar{z}^n$ ($z=x+iy$, $\bar{z}=x-iy$) belongs to a class $\mathcal{C}$ and is pseudo-simple in $|z|<\rho$ then $|C_m(r)|\leqq k(m)r^{m-1}C_1(r)$ for $0<r<\rho$, $m=2, 3, 4, \cdots$. Here $k(m)$ are fixed positive numbers ($k(2)=2$, $k(3)=3$, $k(4)=4.2858$, $k(5)=5.9158$, and so on). Furthermore $C_m(r)=\sum_{n=0}^{\infty}\sum_{\nu=0}^{n}A_{\nu 0}\Gamma(\nu+1)H(m, n, \nu)r^n/\pi^{1/2}\Gamma(\nu+1/2)$ where $H(m, n, \nu)$ are independent of the particular solution u but depend on the class $\mathcal{C}$. (Received January 26, 1946.)

66. Paul Civin: *Polynomial dominants.*

Let $f(x)$ be a continuous function of period 2π, and let $F_n(x)$ be a trigonometric polynomial of degree n which dominates $f(x)$ at the points $x_j^n=2j\pi/(2n+1)$ for $j=0, \cdots, 2n$, that is, $F_n(x_j^n)\geqq|f(x_j^n)|$. Relations of inequality are obtained between various integral means of the functions $f(x)$ and $F_n(x)$. (Received December 26, 1945.)

67. Evelyn Frank: *The location of the zeros of polynomials with complex coefficients.*

This paper develops a rational process for obtaining by successive approximation the zeros of polynomials $P(z)$ with complex coefficients. The process depends upon: (1) If w' is a pure imaginary zero (∞ or 0 included) of $f_r(w)=P(r(1-w)/(1+w))$, $r>0$, then $z'=r(1-w')/(1+w')$ is a zero of $P(z)$. (2) The pure imaginary zeros w' of $P_r(w)=(1+w)^nf_r(w)$ are precisely the zeros of $D_r(w)$, greatest common divisor of $P_r(w)$ and $Q_r(w)=[P_r(w)\pm\bar{P}_r(-w)]/2$ (plus if degree of $P_r(w)$ odd, minus if even; $\bar{P}_r(w)$ obtained by replacing coefficients of $P_r(w)$ by their complex conjugates). (3) By the euclidean algorithm for determining $D_r(w)$, a continued fraction expansion for $Q_r(w)/P_r(w)$ results, in which the number of negative coefficients $c_p(r)$ is precisely the number of zeros of $P(z)$ within $|z|=r$ (Frank, *On the zeros of polynomials with complex coefficients*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 144–157). (4) Hence, if r is varied and changes in signs of the $c_p(r)$ noted, the values of the moduli of the zeros of $P(z)$ can be approximated as closely as desired and simultaneously the corresponding approximation to the zeros themselves can be obtained from the last quotient. The computation can be reduced to synthetic divisions and cross-multiplication processes. (Received December 17, 1945.)

68. R. E. Fullerton: *Compactness in certain Lebesgue spaces.*

Let m be a completely additive measure function defined over a σ-field $\mathcal{J}$ of subsets of a set R. $A(R)$ and $V^p(R)$ will denote the Banach spaces of completely additive set functions which are respectively absolutely continuous and of bounded p-variation over $\mathcal{J}$. Furthermore let $\mathcal{J}$ be compact in the topology defined by the metric $(e_1, e_2) = M[(e_1 \cup e_2) - (e_1 \cap e_2)]$; $e_1, e_2 \in \mathcal{J}$. It is shown that a necessary and sufficient condition that a bounded set in $A(R)$ or $V^p(R)$ be compact is that the function of the set be equi-absolutely continuous in the norm, i. e., if $G_1 \subset A(R)$ $\lim_{m(E)\to 0} \operatorname{var}_E f(e) = 0$ uniformly for $f \in G_1$ and if $G_2 \subset V^p(R)$ $\lim_{m(E)\to 0} [p-\operatorname{var}_E f(e)] = 0$ uniformly for $f \in G_2$. As a corollary it is shown that weak compactness and compactness of sets in the Lebesgue space $L(R)$ are equivalent. In $L^p(R)$ weak compactness and equi-absolute continuity are necessary and sufficient for compactness. Further corollaries give the known condition for compactness in l^p (Cohen and Dunford, Duke Math. J. vol. 3) and the equivalence of weak compactness and compactness in l. (Received January 18, 1946.)

69. Leonard Greenstone: *Mapping by anaytic functions of multiply connected domains.*

Given an arc $\mathfrak{a}$ which lies entirely in some finitely connected domain $\mathfrak{G}$, and let $\mathfrak{G}$ be mapped conformally on $\mathfrak{B}$ so that $\mathfrak{a}$ is mapped into $\mathfrak{p} \subset \mathfrak{B}$. Suppose that the boundary of $\mathfrak{B}$, $\mathfrak{b}$, is sufficiently smooth so that (1) a circle of radius τ can roll freely along $\mathfrak{b}$ for some sufficiently large segment $\mathfrak{b}^* \subset \mathfrak{b}$, and (2) no point of $\mathfrak{p}$ is further from $\mathfrak{b}$ than a distance η, then there exists a constant c, which is independent of $\mathfrak{B}$, such that if $L(\mathfrak{p})$ is the Euclidean length of $\mathfrak{p}$, $L(\mathfrak{p}) \leqq 2\pi^{1/2}\eta c(1+\eta/2\tau)$. c is the non-Euclidean length of $\mathfrak{a}$, that is, $c = \int_{\mathfrak{a}} (K_{\mathfrak{G}}(z,\bar{z}))^{1/2} |dz|$, $K_{\mathfrak{G}}$ being the kernel function of $\mathfrak{G}$ (Bergman, *Partial differential equations, advanced topics*, Brown University, 1941). If, instead of (1), it is only assumed that for at least one point of $\mathfrak{b}^*$ it is possible to place a circle of radius τ tangent to $\mathfrak{b}$ and which lies outside $\mathfrak{B}$, then if (2) is retained and h is defined as the Euclidean distance between the ends of $\mathfrak{p}$, $L(\mathfrak{p}) \leqq 2\pi^{1/2}c(\eta + (\eta^2+h^2)/2\tau)$. Let $\zeta \in \mathfrak{b}$, $\zeta = \xi + i\eta$, then if $f(z)$ is a function analytic in $\mathfrak{B}$, whose real part $f_1(x, y)$ and whose derivatives are single valued in $\mathfrak{B}$, $f(z) = i\int_{\mathfrak{b}} f_1(\xi, \eta)(\int^z K_{\mathfrak{B}}(z_1, \bar{\zeta})dz_1)d\bar{\zeta} + c^*$, c^* constant. This formula identifies the second derivative of the Green's function for $\mathfrak{B}$ with the kernel function for $\mathfrak{B}$ multiplied by -2π. (Received January 12, 1946.)

70. J. D. Hill: *Summability of the sequence of Rademacher functions.*

G. G. Lorentz has recently shown (Math. Zeit. vol. 49 (1944) pp. 724–733) that if the real sequence $\{s_k\}$ is bounded, then the sequence $\{s_k R_k(y)\}$ is summable to zero a.e. by the Cesàro method (C, α) for each $\alpha > 0$, where $R_k(y)$ is the kth Rademacher function. By means of results given recently by the author (Ann. of Math. vol. 46 (1945) pp. 556–562) this note points out generalizations of the Lorentz result to other methods of summability, on the one hand, and to classes of sequences more general than bounded ones, on the other. Further results of a related character are also established. (Received December 28, 1945.)

71. Walter Kohn: *Distortion by simple functions.* Preliminary report.

The strip-method of Groetsch (Berichte der Saechsischen Akademie der Wissenschaften vol. 80 (1927) p. 367) is used to obtain the following properties of simple functions of the type $w = z + a_2 z^2 + \cdots$ and convergent in $|z| < 1$. (I) Denote the region $|z| < 1$ by D and its map by Δ. Let $\rho(\psi)$ be the shortest distance on arg $(w) = \psi$ from $w = 0$ to the boundary of Δ. Then $\int_0^\pi \{\rho(\psi) + \rho(\pi+\psi)\} d\psi \geqq 2\pi$, so that the average length of the segments through $w = 0$ is greater than or equal to 2. (II) Inequalities are obtained for the maps of circular coronas. In particular, as the inner circle shrinks to a point, there results: $\int_D\{|w'(z)|^2/|w(z)|^2-1/|z|^2\}da \geqq 0$, where da is the element of area in D. (III) For bounded functions, $|w| \leqq M$, $|z|<1$, the distortion theorem $r/(1+r)^2 \leqq |w| \leqq r/(1-r)^2$, $|z|=r$, is improved. One obtains for the variation of $|w(z)|$ on $|z|=r$: $(M/2)\{P(r)-(P(r)^2-4)^{1/2}\} \leqq |w|_{\min} \leqq r$; $r \leqq |w|_{\max} \leqq (M/2) \cdot \{P(-r)-(P(-r)^2-4)^{1/2}\}$ where $P(r)=M(r+1/r)+2(M-1)$. These bounds are exact. (Received January 19, 1946.)

72. Harry Pollard: *Integral transforms.*

Integral transformations of L^2 with kernels of type $H(s \pm t)$ are studied with respect to (i) spectral properties, (ii) inversion formulas, (iii) the iterated transformations. By use of methods of Carleman and Stone one can obtain new inversion formulas for the Weierstrass and iterated Laplace transformations. Of particular interest is the resolution of the identity of the Laplace transform, since it has apparently escaped the attention of workers in the field that this transform is a bounded self-adjoint transformation of Hilbert space with a continuous spectrum. (Received January 19, 1946.)

73. Harry Pollard: *Necessary and sufficient conditions for the representation of an analytic function as a Laguerre series.* Preliminary report.

Necessary and sufficient conditions are obtained for a function analytic in a parabola to be representable by a complex Laguerre series. This complements the corresponding result of Hille for Hermite series (Trans. Amer. Math. Soc. vol. 47 (1940) p. 80). The proof of the necessity parallels that of Hille, although the technical details are more complicated. The sufficiency proof is of a different kind from Hille's, depending on a simple though apparently new formula connecting the Laguerre and Hermite polynomials. (Received January 19, 1946.)

74. Menahem Schiffer: *On the kernel function of an orthonormal system.*

Every function $f(z)$, analytic in a domain D and with finite $\iint_D |f(z)|^2 dxdy$, satisfies the integral equation $(*) f(z)=\iint_D f(\xi)\mathrm{K}(z,\bar{\xi})d\xi d\eta$ where $\mathrm{K}(z,\bar{\xi})$ is the kernel function defined by Bergman (*Über die Kernfunktion eines Bereiches*, J. Reine Angew. Math. vol. 169 (1933) pp. 1–42). The kernel function is shown to be determined uniquely by this property. Then if $g(z,\xi)$ denotes Green's function of D, $-(2/\pi)\partial^2 g(z,\xi)/\partial z\partial\bar{\xi}$ is proved, by integration by parts, to have this characteristic property and to coincide, therefore, with $\mathrm{K}(z,\bar{\xi})$. From Hadamard's variation formula for Green's function the variational formula $\delta\mathrm{K}(z,\bar{\xi})=\oint \mathrm{K}(z,\bar{t})\mathrm{K}(t,\bar{\xi})\delta n ds$ is derived. If $\phi(z,\xi)$ is an analytic function of z which maps D upon the unit circle slit along concentric circular arcs such that $z=\xi$ corresponds to the center, and $\gamma(z,\xi) = -\log|\phi(z,\xi)|$, the kernel $\mathrm{K}^*(z,\bar{\xi})=-(2/\pi)\partial^2\gamma(z,\xi)/\partial z\partial\bar{\xi}$ satisfies equation (*) with respect to every $f(z)$ which is square integrable in D. $\mathrm{K}^*(\xi,\xi)^{-1}$ is the minimum value of all integrals $\iint_D |f'(z)|^2 dxdy$ where $f(z)$ is analytic in D and $f'(\xi)=1$. If D is mapped by univalent functions $g(z)=1/(z-\xi)+a_0+a_1(z-\xi)+\cdots$ upon an infinite domain Δ, the complement Δ^+ of Δ has an area not greater than $\pi^2\mathrm{K}^*(\xi,\bar{\xi})$. (Received January 14, 1946.)

75. I. E. Segal: *Topological groups in which multiplication on one side is differentiable.*

It is shown that if left multiplication is of class C^1 on a topological group which is a manifold of class C^1, then right multiplication is likewise of class C^1. The proof strongly utilizes the existence of Haar measure. The method does not involve introduction of canonical coordinates, and is logically independent of the theory of Lie groups. Combining the foregoing result with either a result of P. A. Smith (Ann. of Math. vol. 44 (1943) pp. 480–513) or a result of G. Birkhoff (Trans. Amer. Math. Soc. vol. 43 (1938) pp. 61–101), it follows that if a group has a neighborhood of the identity which is a manifold of class C^1 and on which left multiplication is of class C^1, then it is a Lie group. Previously the weakest condition which assured that a group be a Lie group was that due to Smith and required in addition to the preceding that right multiplication satisfy a Lipschitz condition. (Received January 21, 1946.)

76. H. J. Zimmerberg: *A class of definite integral systems.*

This paper treats a vector integral system of the form $y(x)=\lambda\int_a^b K(x, t)y(t)dt$, where the matrix $K(x, t)=H(x, t)S(t)$ and the discontinuities of the elements of the matrix $H(x, t)$ are regularly distributed, under the following conditions: (1) the matrix $S(x)$ is hermitian on ab; (2) the matrix $K_1(x, t)\equiv S(x)K(x, t)=K_1^*(t, x)$ on $a\leqq x, t\leqq b$, the * denoting the conjugate transpose operation; (3) the hermitian functional $J(g)\equiv\int_a^b\int_a^b g^*(x)K_1(x, t)g(t)dxdt\geqq 0$ for arbitrary vectors $g(x)$ whose components are continuous on ab. These systems differ from the definitely self-conjugate adjoint integral systems of Wilkins (Duke Math. J. vol. 11 (1944) pp. 155–166) in that the above condition (3) replaces a corresponding definiteness assumption on the matrix $S(x)$. Such systems, which include the integral system to which an H-definitely self-conjugate adjoint differential system (Reid, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 381–425) is equivalent, are shown to possess fundamental properties similar to those previously established for definitely self-conjugate adjoint integral systems. Furthermore, it is shown that there is no restriction on the character of the characteristic solutions or the corresponding characteristic values in assuming that an integral system of the above form satisfies conditions (1) and (2). (Received December 26, 1945.)

77. H. J. Zimmerberg: *On a self-adjoint differential system of even order.*

The results of Reid (Trans. Amer. Math. Soc. vol. 52 (1942) §11) are extended to a self-adjoint system $F(u)=\lambda G(u)$, $U_\sigma(u;\ \lambda)\equiv U_\sigma^0(u)+\lambda U_\sigma(u)=0$ $(\sigma=1, \cdots, 2n)$, where $F(u)$ and $G(u)$ are differential operators of the form $F(u)=\sum_{\nu=0}^{n}(f_\nu(x)u^{(\nu)})^{(\nu)}$, $G(u)=\sum_{\mu=0}^{n-1}(g_\mu(x)u^{(\mu)})^{(\mu)}$, $f_n(x)\neq 0$, while for arbitrary values of λ the $U_\sigma(u;\ \lambda)$ are independent linear forms in the end values of $u, u', \cdots, u^{(2n-1)}$ at $x=a$ and $x=b$ for which $U^1(u)$ involves only the end values of $u, u', \cdots, u^{(n-1)}$. It is shown that such systems are equivalent to certain types of boundary value problems associated with the second variation of an isoperimetric problem of Bolza in the calculus of variations. Various assumptions of definiteness for these systems are also considered. (Received December 3, 1945.)

78. M. A. Zorn: *Derivatives and Fréchet differentials.*

It is shown that, for complex Banach spaces, the continuity of the Gâteaux differential in its increment variable makes the G-differential a Fréchet differential. Using a notion which has also been introduced by A. D. Michal this may be re-expressed in the form: If a function on an open set of a complex Banach space to a complex Banach space has a derivative, it possesses a Fréchet differential. (Received January 22, 1946.)

Applied Mathematics

79. R. J. Duffin: *Nonlinear networks.* II.

A system of n nonlinear differential equations is studied and shown to have a unique asymptotic solution; that is, all solutions approach each other as the independent variable becomes infinite. The interest of these equations is that they describe the forced vibration of electrical networks. Consider an arbitrary linear network of inductors, resistors, and capacitors which has no undamped free modes of vibration. A given impressed force may give rise to more than one response but as time goes on there is a unique association between impressed force and response. This, of course, is well known. The main result of this note states that if the linear resistors of such a network are replaced by quasi-linear resistors then there still is this unique asymptotic association. A quasi-linear resistor is one in which the potential drop across it and the current through it are increasing functions of one another. No other sort of nonlinearity besides this type of nonlinear damping is considered. The proof is made to rest solely on well known properties of the Laplace transform and Hermitian forms. (Received January 19, 1946.)

80. Herbert Jehle: *Transformation of hydrodynamical equations of stellar dynamics.*

In abstract 51-9-170, the author pointed out a transformation of continuity equation and Bernoulli equation into a Schroedinger equation. The presence of $\bar{\sigma}$ (replacing $\hbar/m$ of wave mechanics) implies no modification of classical equations of motion, but a statement about residual velocities or "pressure function." Assume that the distribution (numbers and intensities) of excited ψ_{nlm} states (n goes up to about 10^5 for the author's choice of $\bar{\sigma}$) corresponds statistically to the distribution (in numbers and masses) of statistically independent elements of a system. It is known that if all stationary ψ_{nlm} states are filled up to a certain frequency limit with one element (particle) per state there will be an average of one element per phase space volume $(2\pi\bar{\sigma})^3$ for the inner regions. The above assumption is therefore equivalent to the assumption of an upper limit for the expectation value of density (of numbers) of statistically independent elements in six-dimensional phase space. This is a plausible assumption for systems close to statistical equilibrium; it means that too great densities in position space without large residual velocities cause aggregations of formerly independent masses into larger independent units. (Received January 21, 1946.)

81. R. S. Phillips: *rms error criterion in servo system design.*

A servomechanism is required to follow a signal from a knowledge of only the error in following. This error signal is usually a mixture of the true following error and some sort of random disturbance. The servo must make a compromise between following the original signal and not following the noise. This paper presents a method by which this compromise can be made. Assuming that the spectra of the signal and noise are known, one can then determine that servo system which minimizes the rms error in following. Actually the paper limits itself to determining the best values of control parameters when the type of control is given. It is assumed that the servo